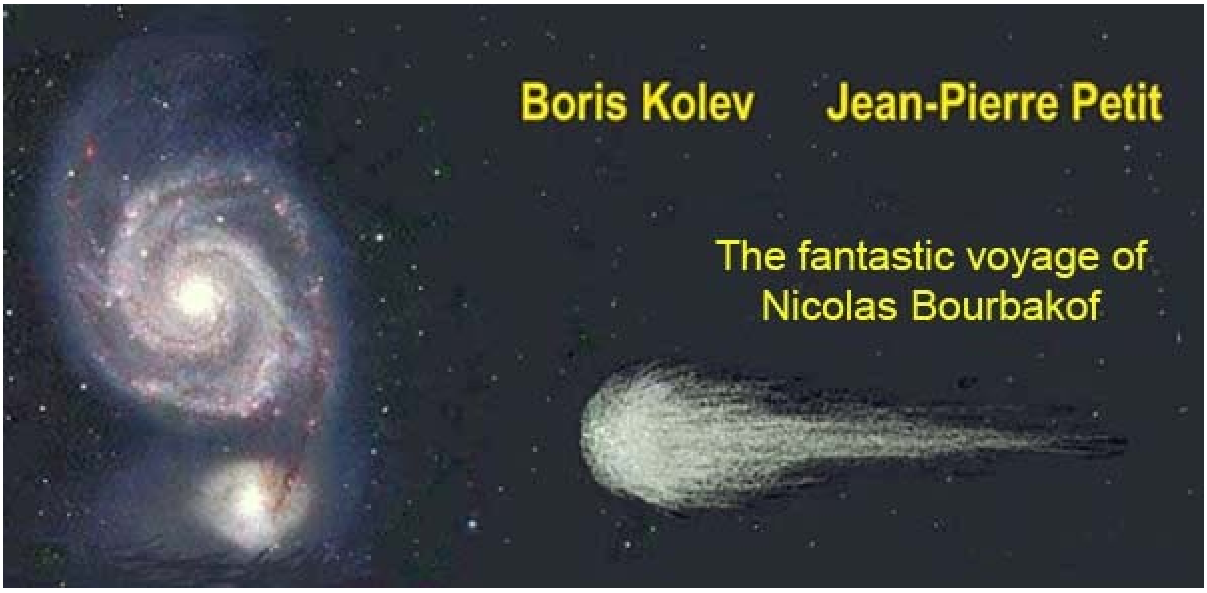
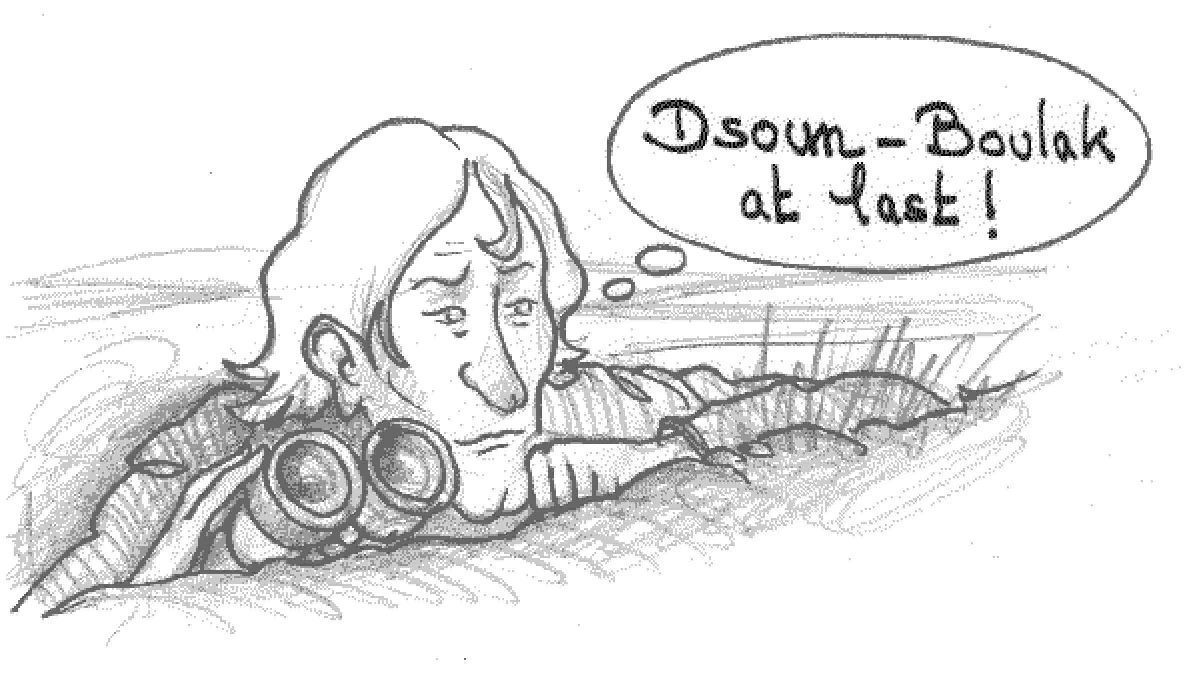


Nicolas Bourbakof is going to meet Hubert de la Boissinière team in Dsoun Boulak, in Central Asia, which is buildings a strange ship, fuelled in part by a strange propulsor supplied by a certain Jacobson. Everyone takes off. In any direction outside the Solar System. By accelerating to a half g the ship can pass beyond Pluto in 17 days. Bourbakof makes friends on board, Fowler, a physician who had fled Princeton, Turyshev, a biologist. Boissinière waits impatiently for a message from Jacobson, who will send instructions for this unprecedented voyage. Unfortunately a solar eruption interrupts reception. The members of the expedition are "flying blind". They only heard the beginning of the message when they learnt that there was "sufficient mathematic knowledge in Bourbakof's head to fly the mission". But what knowledge Boissinière asked himself? How could they go and look into his head. What questions should they ask? Boissinière asks Bourbakof to give seminars to the crew. A result of these is the Lagrange equation for a soap bubble problem. Turyshev thinks to himself that he should have been a mathematician. Suddenly Picard, the ship's astronomer, detects an enormous field of ice blocks directly in line with the ships trajectory and it is moving at 8000 km/h. A that speed it is impossible to take avoiding action. It is the debris of the 11th plane, disintegrated by a tidal effect and moving towards Earth. It is the reason for this last chance mission. Meanwhile, on Earth, Jacobson and his "friends" are going to try to destroy the swarm of comets which won't be long in arriving otherwise. Boissinière remembers a blue envelope that Jacobson left him: "to be opened when you've left the Solar System". He opens it ...

Dsoun - Boulak

Nicolas Bourbakof's mouth was dry. He had decided to leave this shameful regime definitively. Despite the science police, the terrible epistemocops who gave no quarter, he had decided to try to rejoin "those who have chosen to go". He could no longer put up with imagining that the all of the science police were after him, relentlessly, in a world that had to be "scientifically correct" and "mathematically correct" everywhere. A world which reminded him of the film "Brazil" without horizons, without crests of waves, where words of love and friendship seemed to have lost all meaning. Words that had been replaced by "cooptation", "integration". There the future was already traced out, "knowledge motorways" which led nowhere. An enormous circular structure set up by the new scientific rationalists where one went round and round and was fully aware of doing so. Even as a child when a member of the Young Scientists he had never felt easy in that world. The Collective ran everything from the birth to death of each individual, taught from childhood that it was everyone's small suffering that brought about the great general good, so that the more one had small particular woes, the better it was for the "best of all possible scientific worlds". Identity was the guarantee of stability and immobility was the greatest factor for progress. This guideline set by the Party, had influence physicians even before Bourbakof had decided to flee. One of them had even given his caution to a thesis entitled "The Evolution of Static States".

The journey was terrible. He had had to hide everywhere and cover thousands of kilometres on foot or by hanging on to train wagons. The only thing he knew was the name of the place he was heading to.



At the scruffy village on the edge of Outer Mongolia, beyond the yurts and corrals of small horses whose blonde manes floated in the wind, he could see modern tents and enormous hangars. Like a drunken man he staggered to the village until suddenly he was called by an enormous bald-headed man:

- Bourbakof! By Jove! If only I had known that a fellow like you were coming to join us.... Come and refresh yourself, you look all in.

- I could sleep for a week. Who are you?

- Hubert de la Boissinière, director of the project.

- Project?

- Yes, I'll explain but for the moment you seem to be too tired to have a serious conversation. Come, we'll look after you.

De Boissinière led Bourbakof into a tent.

- You have whisky here!

- No! It's kulik, a local thing. But it will do you good.

- Boissinière... that rings a bell. Aren't you a specialist in MHD?

- Exact.

- So what is this ... project?

- That's what you came for isn't it?

- Of course. Maths is finished for me. I'm ready for anything ... even physics.

- You are earnest! We'll be needing you. Don't leave your precious knowledge of geometrics on this planet, doomed in the near future to an extension of bacteriological war.

- And how in heaven can that knowledge be useful to you?

- Don't you know where we're going?

- Where?

- Exactly. We don't know either. Geometry is the science of the unforeseen isn't it?

- Hm ...

There was a long silence. Boissinière laughed.

- Your colour is coming back in front of my eyes. Cigar?

- A local thing?

- No, these come from Havana.

Bourbakof appreciated the sensation of the Havana smoke and the whisky.

- In short, we are leaving Earth.

- Exact.

- By MHD?

- Yes and no. It all depends on some assistance which we should be getting any day now. The project managed to come together because Jacobson succeeded in crossing the lines with the Iliouchin and its baby.

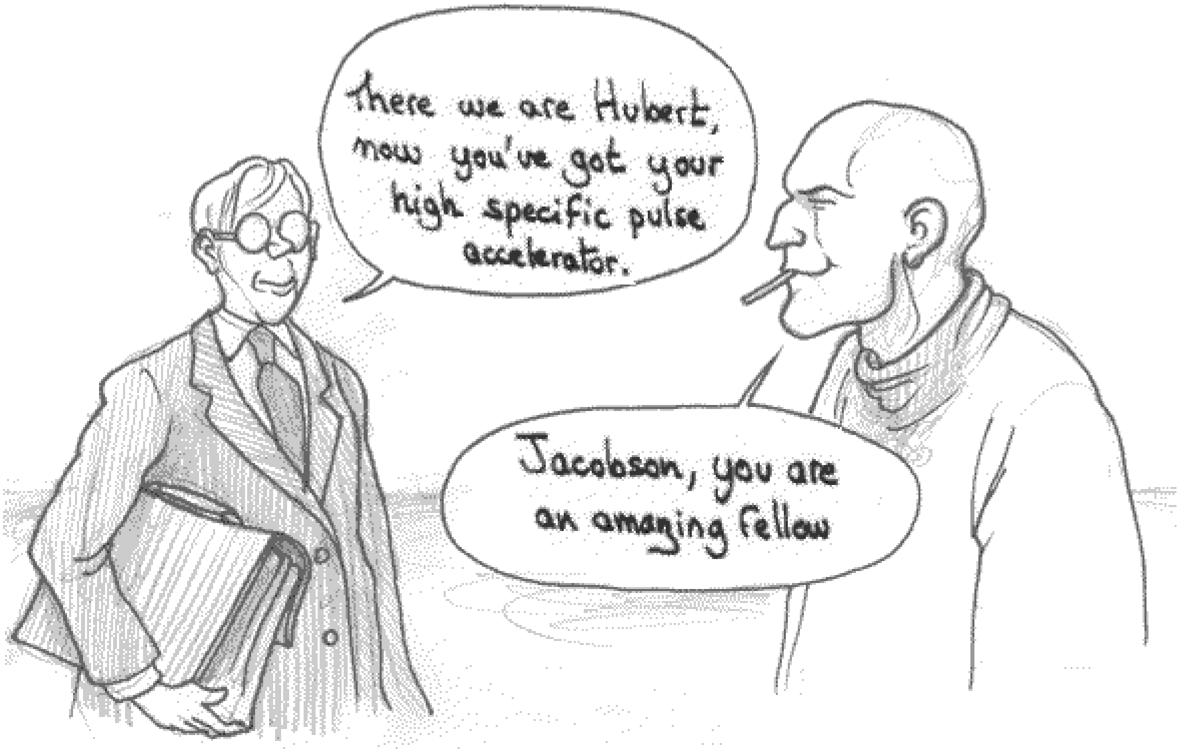
- And what is that??

- I told you, it's the key to the success of the operation. But Jacobson didn't want to tell me any more. Maybe he doesn't even know himself.

Bourbakof didn't reply, he had fallen asleep with his head on the table. Boissinière got him onto a camp bed and removed his shoes. He slept for forty-eight hours and snored like a donkey.

The Iliouchin landed in a cloud of dust. The members of the little colony ran towards the door bringing with them just an ordinary ladder for the passengers to use to descend. Jacobson was the first to appear.

- Formidable, you managed to get here, Boissinière exclaimed.



The rear ramp of the Iliouchin was lowered. Workmen undid the fixing cords to remove and very carefully descend an enormous package that filled almost all the cargo hold.

- Good, now you have everything you need. I've put all the documents you'll need to help you operate this thing in this briefcase.

- Do we know how it works?

- No, and they've asked us not to open it. There are two things that will interest you. The first is the electric power output. It’s got everything, direct, LF and HF, at three gigahertz.

- Well...

- At the end, the nozzle. It should give you more than enough thrust, but only beyond the Earth's atmosphere. Taking into account the structural resistance of the ship, I would advise limiting acceleration when cruising. For takeoff and crossing the atmospheric layer you'll need to push against air with MHD.

- If there is the electric power, we'll get the ship out, no problem.

- I'll trust you on that. Excuse me, I've got to go. "Professor Noah", goodbye and good luck.

- Aren't you stopping for a while?

- No, I have to get back to area 51.

- So you don't want to tell me what you are getting up to over there?

- I could, but it would be long and I don't have the time. In any case, once you've been on your way for a while you'll understand completely.

- OK, I won't insist. We'll have to get to work as soon as possible. Thanks anyway. And thank your ... friends too.

- I won't fail, but that is something they are not very used to.

Jacobson had already turned and reached the plane with a rapid step. As he came out of his tent, Bourbakof saw the Iliouchin rise from the ground at the end of the runway. He rejoined Boissinière who was organizing the towing of the enormous engine, covered by a tarpaulin, towards the main hangar.

- There we are, now it is up to us. We have our cruising propulsor.

- Do you mean that this thing will enable you to leave Earth?

- With you, yes, unless you've changed your mind.

- Certainly not.

The following weeks were dedicated to adapting the propulsor-generator group to the discoid MHD ship that Boissinière had built from spare parts and brought to Dsoun-Boulak in great secret.

- You remind me of Captain Nemo and his Nautilus.

- I can see your point, except that instead of exploring the seas we are going into the Cosmos.

- Direction?

- The constellation of Virgo, there or elsewhere....

Departure

After this first contact with Hubert de la Boissinière and during the wait for the ship too take off, Bourbakof took his quarters in the centre. It was minimum comfort: a simple tent, but he made do. Luxury had never been his thing. A table, paper and a pen were enough for him to create a whole universe. Boissinière, whose face showed signs of tiredness, came to get him one day and took him to see the ship stored in the giant hangar number 3. There wasn't enough space to be able to step back and see its exact shape but it seemed enormous. Maybe two hundred metres in diameter. Technicians were working on the lower part installing the propulsor that Jacobson had brought. It had seemed enormous in relation to the Iliouchin cargo plane that had brought it there but now seemed ridiculously small in relation to the spaceship it was meant to propel.

- This will be our cruising propulsor, Boissinière commented, which will only be used when we are outside the Earth's atmosphere.

- How are you going to get this sort of Noah's Ark off the ground?

- I'll explain when we have the time. For the moment I have too much to do.

- I understand, but how did you get financing for a project such as this? It seems mad....

- Everything came from Jacobson.

- I thought he was attached to zone 51?

- Correct. Zone 51 is in Nevada, in the United States.

- A state within the state, as I see it.

- To be honest, you'd have to be pretty clever to know who the real bosses of the place were. I couldn't get a word out of Sven on the subject.

- As far as I could find out from the technicians he was a specialist in power lasers.

- Correct. One day Jacobson said to me "would you be interested in a project that will have no limit on finance?". I replied "it depends on the finality of the project". Jacobson put his hands on my shoulders and looked me in the eyes before saying "I expected a reply such as this from you". He gave me a rough outline of an enterprise that seemed to me to be completely mad. There were supposed to be two teams, one would stay and one would leave.

- And here it is the team that is going?

- Indeed. But leaving is easier to say than do. You have seen the ridiculous pile of rust that the ISS has become, the International Space Station that nobody can afford to keep up. At first I thought that Jacobson would propose something like the installation of a departure base on the Moon. All he would say was "that is not anticipated" and smile. Personally, if I was given the necessary electric power, I would do my best to get anything off the surface of the Earth and even to get it out of the attraction of our planet but even at eleven kilometres a second, you don't go very far.

- So Jacobson offered the electrical generator?

- Yes, but he asked me to not ask questions on the subject. As well as that we'll receive as much superconducting wire as we want. It is capable of resisting absolutely indecent temperatures. I designed the ship and left a space for this thing, he told me the size, so that it could be joined to the structure later and all the electrical connections made subsequently. We are about to undertake this integration. With the time required for tests we should be ready to cast off in a week or two.

- That still doesn't tell me who is financing it.

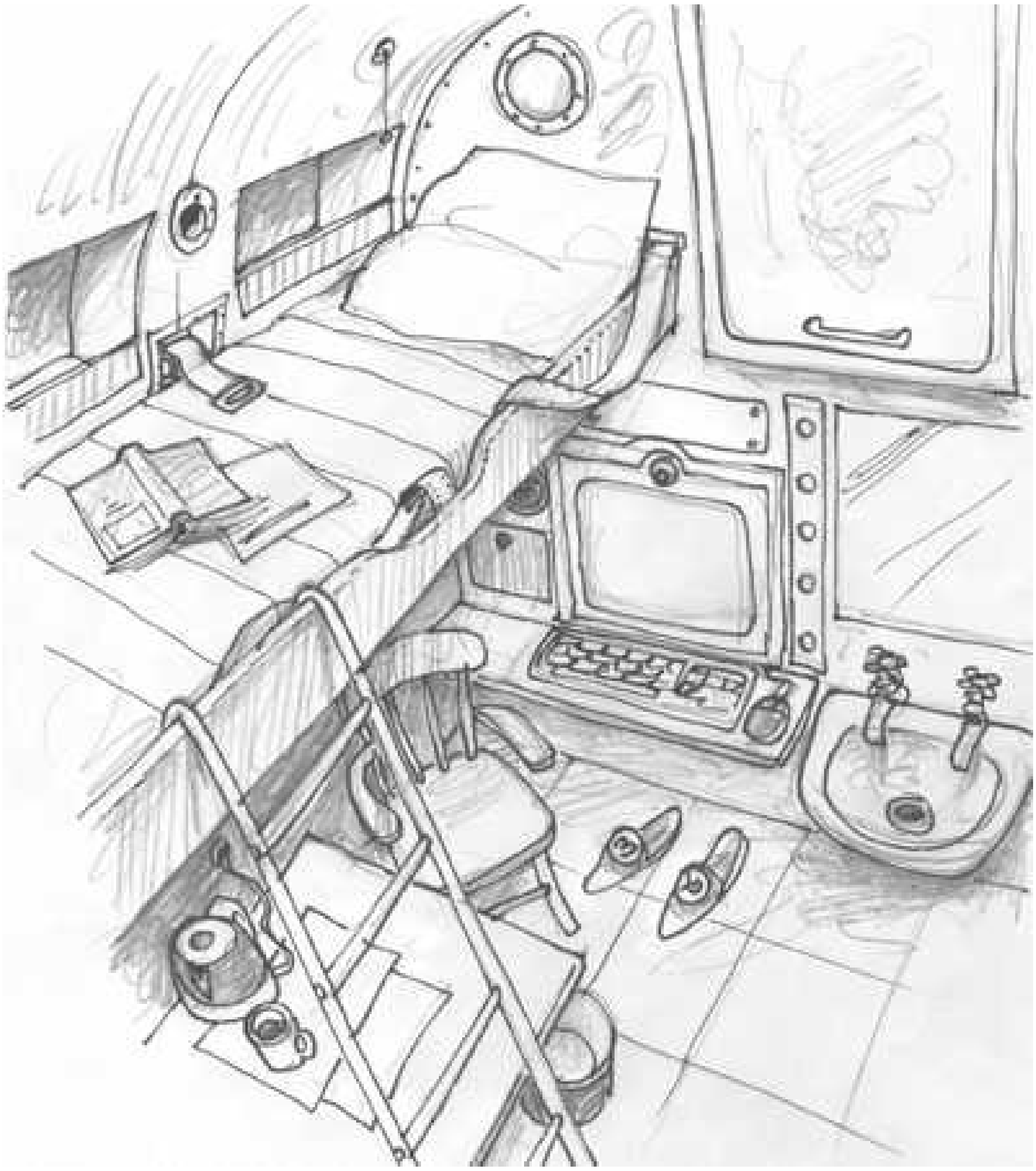
Bourbakof looked around.

- Nothing was done here. Jacobson came by every two months and took the plans. The next time he came the Iliouchin brought the elements to be assembled. Everything had to be designed as a kit. Elements that were lighter but even bigger were brought in by powerful helicopters.

- You mean that the Russians helped all along?

- I told you: Jacobson told me not to ask questions, I didn't ask questions.

The day of departure arrived. The small blond-manned horses were released and the technicians chased them in 4WDs until they were far enough away from the departure zone. The hangar was dismantled and the ship appeared. It looked like two enormous upside-down soup plates balanced on three telescopic legs. The contrast was strange between this futuristic object and those that were going in it. Some of them, who had only finished the final touches late the previous evening, hadn't even had time to shave. Their personal luggage was strictly limited. Boissinière wore a white coat. Bourbakof meekly took his place in the column lining up to go on board. Everyone was quiet. One could feel that they were touched with a sense of the seriousness of the moment but no speeches were made. They filed into the ship that was all. The badge of each person was checked on entry and each received a simple card showing the number of their cabin. One could imagine being on a Cunard liner.



Everyone had an individual cabin with minimal comfort: bed, desk, shower and WC. Boissinière appeared on the video screen.

- OK, I must ask you all to lie on your beds and attach the safety harness. It would be best if everyone was in place as quickly as possible. We'll let you know when we are going.

Bourbakof obeyed. Fresh air was coming in rather noisily from above the bed through a simple hole. The wait continued for a good hour. He went over the numerous events of the past months and years in his mind. As he lay on the bed he was gradually taken over with a sense of ineluctability until the voice of Boissinière brought him out of his dream:

- OK. I suppose that everyone is ready. Let's go.

There was no vibration, no noise, nothing. Just a constant acceleration that could hardly be felt. After twenty minutes or so Bourbakof had the disagreeable impression that the ship was falling back to Earth. It was awful, like falling down a well. Floating in his harness above the mattress he closed his eyes and thought to himself "it hasn't worked, we're going to crash". In fact the ship had already left the Earth's atmosphere and had simply followed a ballistic trajectory for a dozen or so, interminable, seconds. When the cruising propulsor started, Bourbakof suddenly fell back on his bed. Boissinière's voice came from the loudspeaker:

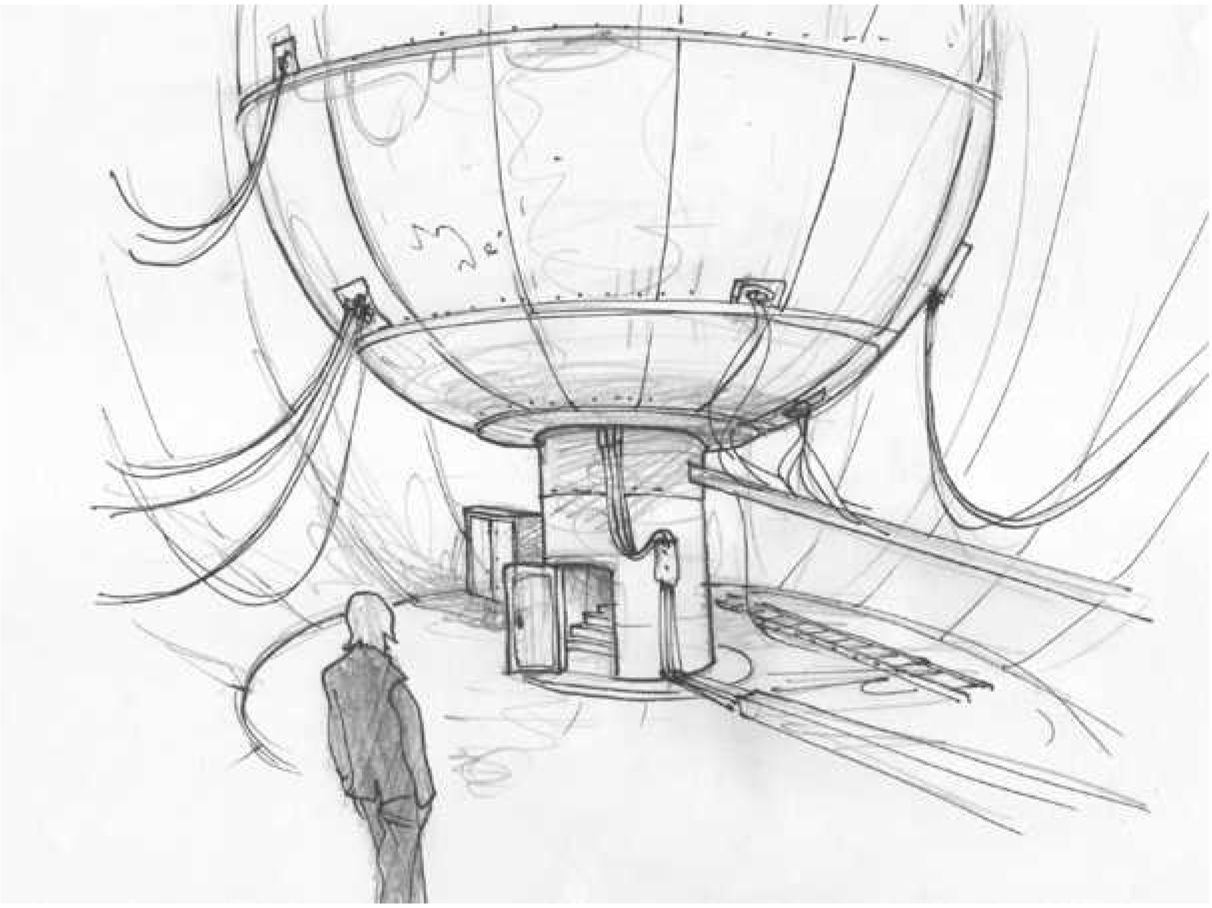
- Takeoff succeeded. You can unfasten your belts.

The impression was strange. The acceleration has been less than one g. Everyone had to learn how to walk with the reduced gravity. Going downstairs without falling was the most difficult. During the moments following the injection onto the trajectory the vessel was filled with sorts of ghosts that seemed to move on tiptoe with great precaution. Boissinière's voice came over the portable that Bourbakof, like everyone else, had clipped to his shirt pocket.

- You can join me on the bridge. It is definitely worth a look. Just put on your headphones and follow the direction of the sound you hear.

Bourbakof did so. By 'instinct he turned his head and moved towards the entrance of a corridor. The system sent out a signal each time that he was required to change direction, it would suddenly come from the right or the left accordingly. At the end of the final corridor he came to a sort of room containing a sphere with many wires attached which led to a sort of cylindrical base, in fact a spiral stairway. Bourbakof heard the voice of Boissinière which had become sepulchral due to a resonance phenomenon:

- Come.



The sphere was completely hollow with a diameter of a dozen or so metres. After climbing the stairs he arrived on a round platform with space for ten or twelve people and surrounded by a protective rail. There were also several consoles installed there. Boissinière was seated at one of them. Intrigued, Bourbakof approached, holding the rail mechanically as he did so. Boissinière’s face lit up in a smile as he pressed a button. At that moment Bourbakof had the surprise of his life and had the impression that they were passengers on the circular platform which had suddenly been projected into empty space. The Sun was visible and lit up part of the Earth's surface. He estimated their altitude to be about a thousand kilometres.

- Nice view huh? I wouldn't want you to leave good old Earth without having a last look.

- But where are we?

- In the sphere of course! That's just a visualization screen. On its internal face there is a liquid crystal imaging system linked to exterior 'windows'. It's better than a porthole don't you think? Here, take this trackball.

Boissinière held out a box containing a plastic sphere, similar to the command systems that used to be found on ancient microcomputers.

- If you move the ball you can change the point of view at will.

In effect, by moving the sphere the "celestial canopy" changed, though not the apparent gravity.

- Oh la la !....

Bourbakof felt sick.

- Be careful, don't overdo it or you'll make us all seasick.

- Of course, the visual signals don't go with the subjective impression because of the artificial gravity we feel.

- Yes, classic. If the scene changes and the signals sent by the internal ear don't match, then boom, seasickness.

Bourbakof gave back the trackball.

- That's enough for me thanks.

- You have the same thing in your cabin with a trackball integrated into the desktop. The computer screen will act like a porthole whose point of view you can modify at will. The advantage in your cabin is that the toilets are never far away.

The experience was nevertheless surprising for Bourbakof who felt as if he had just got off a fairground ride. That evening he preferred going to bed in his cabin rather than having dinner but in the following days he easily found his way to the refectory and the well-filled library which contained a great variety of books. From ten on he spent many hours reading.

The seminar

Boissinière was in his cabin. Three weeks has passed since departure. Acceleration had been stabilized at a half g and at that speed the velocimeter showed that the ship was cruising at 8640 kilometres a second. By then it was crossing the orbit of Jupiter. The spectacle was magnificent. In comparison the earth was just a ridiculous speck. They had tried to enter into radio contact with Jacobson several times since departure but without success. Now, at five light hours from earth, all direct conversation had become impossible. Suddenly a lamp lit up signalling the arrival of a message. He set the equipment to listen mode and pivoted his seat towards the plasma screen on his desk. Jacobson's head appeared.

- At last! Boissinière exclaimed.

- My dear Hubert, I see that you are having a good journey. Your parameters seem OK. I suppose I owe you a few explanations. They'll maybe be the last as in principle, according to the flight plans that we have established, you will soon be arriving at a point when we will lose all contact. You haven't yet left the Solar system so you don't yet have a visual on the thing that motivated our mission, which means that as we did not want to put all our eggs in the same basket, there are two teams: you, and us. You should be wondering what you can do to fill your days on this one way trip that we've offered you. I imagine that you have taken some card games with you and plenty of documents of all sorts to help you kill time. In fact you have a very precise job to do which I shall now explain. For us, you are like a sort of cell that has been projected outside the solar system. Your ship contains two types of information. Firstly, what you are supposed to know and understand, then what is contained in the cylinder our friends gave us which, from its site at the back of your ship, gives you the constant acceleration of a half g. There is no point in trying to open it for even if you did so it seems that you wouldn't be able to understand the how and why of its components.

- We have a sort of "black box" at the back Boissinière thought.

He listened to the rest of the message.

- By accepting this our friends say that they have transgressed what is a very strict law. In effect you have been given means, albeit primitive in comparison with theirs, which allow you to reach other systems than the solar system and where you could create a certain disorder apparently. Let us say that your departure from the nest is a little premature given the state of maturity of the human species, which we could not call extraordinary as I'm sure you'd agree. And I include myself in that. But circumstances have decided otherwise.

- When you leaver the solar system, open the blue envelope that is in the dossier with the instructions I gave you for the equipment I brought in the Iliouchin, the propulsor-generator. You will find additional instructions. As you have embarked on this adventure my friends considered that it was necessary to give us the possibility of making a leap forward in the level of our knowledge. For this, two special passengers have been included in your team. The first is Nicolas Bourbakof, whose escape we discreetly helped. It wasn't easy. Given his craftiness he was almost taken at least three times because of his tricks and his phosphorescent cable. When he arrived safely at Dsoun-Boulak we were quite relieved I can assure you. My friends say that in his brain, whose every synaptic connection they analysed, this fellow has sufficient knowledge to allow you to make the leap. However he doesn't' know this and it is perhaps preferable for the moment that he remain in ignorance. I will explain the second part of the message, what level it concerns and how to get into the brain of the lad and the dossiers to start with.

There is a second fellow on board who we also sent to help you. We decided this without your knowledge. I hope you won't hold that against me but we didn't want to take any risks. Like Bourbakof, he doesn't know why he is with you or his future task. His name is ...

The image broke up. Boissinière adjusted several buttons in an attempt to activate a programme for the restoration of the signal but without result.

- Blasted sunspots!

That was a problem in the middle of reception. He waited for a second message during the following days but got nothing other than silence except for the background noise of solar eruptions.

- It must have completely destroyed the antenna.

He had foreseen everything except that.

- I'm an idiot!

A reserve antenna would have been enough, hidden away in a space somewhere where it would be protected against bombardment during an eruption. He had counted on three parietal antennae, real jewels, but hadn't thought of the possibility of simultaneous destruction by the clouds of plasma emitted during solar eruptions. Now he was cut off from Earth, both in reception and transmission, for a good while.

Boissinière tapped a code number on a keyboard and an image of the inside of Bourbakof's cabin appeared on the scope. Bourbakof lay on the bed reading a treatise on algebraic geometry and listening to Mozart.

- What am I going to do with this animal? .....

Bourbakof replied to Boissinières call. The latter seemed very down. He was at his table and had got out a child's toy, a thing for making soap bubbles.

- Where did you ever get such a thing?

- There is just about everything on this ship, except real whisky alas, otherwise I would have immediately drunk an entire bottle on my own.

- Have you got the blues, Boissinière?

- I don't know how you manage. One gets the impression that we could abandon you for ten years on a desert island with some maths books and you wouldn't even notice the time passing.

- I like mathematics, it relaxes me.

- You're reading that like a comic book!

- True enough. Some pages do indeed contain humour and suspense...

Boissinière raised his eyes to heaven. He had poured the bubble making liquid into a flat basin and had different objects on which to attach an iridescent film. He chose two of them, circles of the same radius on the end of a stick. If he did it right he could make a more or less cylindrical film between the two circles.

Bourbakof intervened:

- It is an interesting variational problem. Did you know that for a given value for the radius of the circles there is a maximum separation distance beyond which the film cannot be maintained and that this distance can be calculated?

Boissinière did the experiment. He distanced the circles from each other and indeed there was a moment when the film broke to leave just two circular membranes, one on each circle.

- And you can calculate this?

- Kid's stuff! It is just a few lines of calculation.

- Could you show us that at a ... seminar?

- Whenever you want.

- OK... I'll publish an announcement. It'll help improve morale among the troops. Especially as we still have a long wait before reaching the edge of the solar system.

The ambience at the seminar was pretty retro. In fact, when Boissinière constructed the ship, all that was needed other than for propulsion had been found in the area. On the end of a declining empire, and by paying cash, in dollars, you could get just about everything.

Someone brought an old blackboard and a box of chalk. Others had found seats and desks. Bourbakof set off with great speed.

One of those present leaned towards Boissinière:

- Hubert, I can't even take notes. I get the impression that your left hand is rubbing out what your right hand has just written.

- Yes, said Bourbakof, white from chalk dust.

Boissinière cleared his throat.

- Very impressive. But I think you don't realize that you are not in a university amphitheatre here. Around you, you have various people, physicians, chemists and a few biologists with a certain experience in mathematics. Your demonstration was impressive but you were going way too fast.

- I write badly I know, I've often been told so, but all this is absolutely elementary!

- I am sure it is but I think the only solution for us to get out alive from one of your seminars is to take everything step by step. My wish is that everyone understands. We don't have your mental agility. !

- But Boissinière, to build such a ship as this you had to use pretty sophisticated knowledge no?

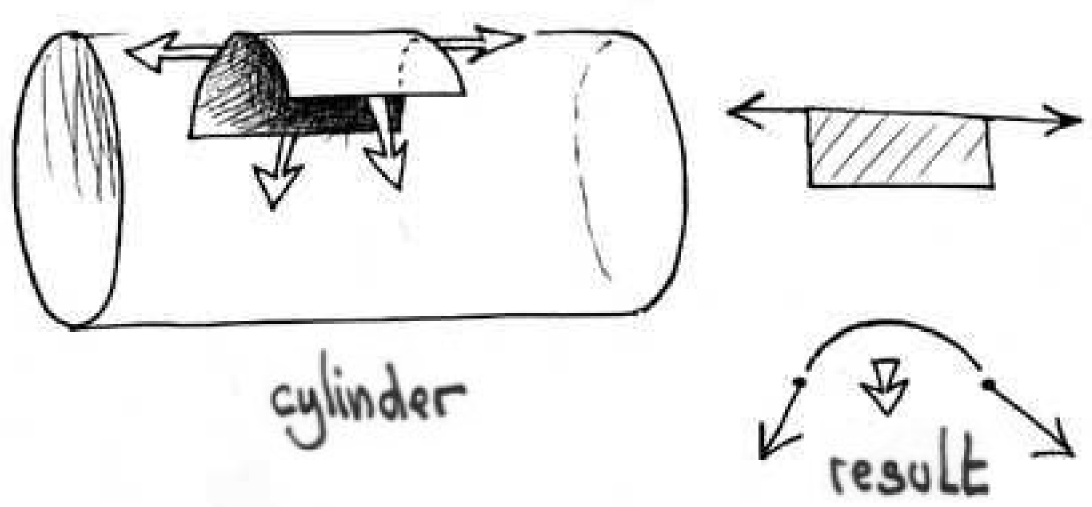
- Yes and no...

An American physician by the name of Fowler burst out laughing:

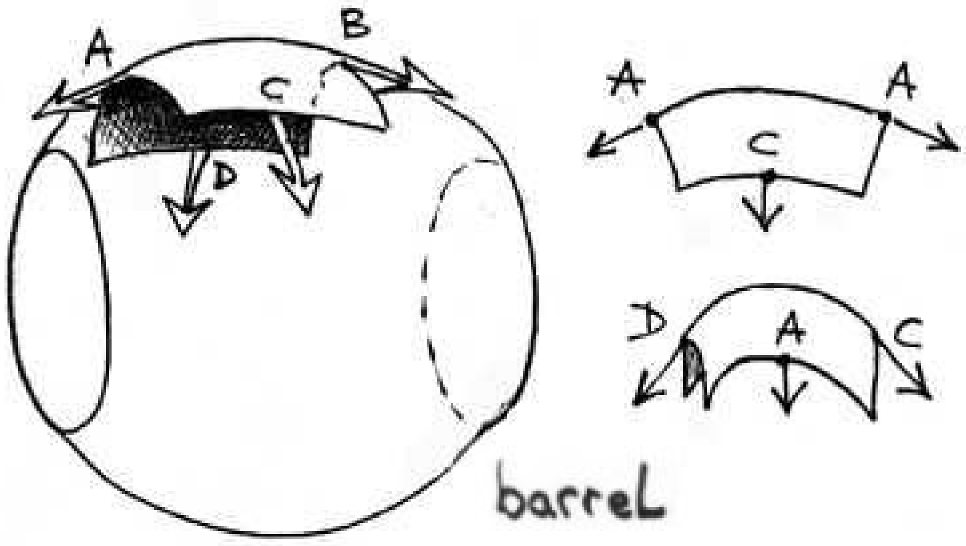
- My dear Bourbakof, you know that in physics it is often more a question of courage rather than of intelligence!

Boissinière moved to the blackboard and tried to rebuild his 'argument.

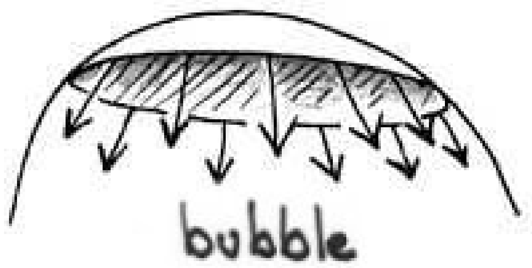
- OK, before Bourbakof's lecture everyone saw the little experiment where a film of soap with symmetry of revolution rested on two coaxial circles with the same radius R. Everyone understood that forces of tension in tangent with the film were involved. We also understood why we couldn't create a cylindrical film between the two circles. If I draw an element of a cylinder I can see that the result of the forces is not nil. In the absence of another force, linked for example to a pressure difference on either side of the film, the cylinder will tend to contract:



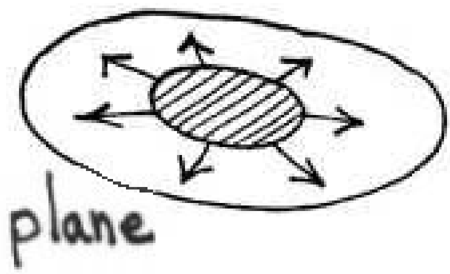
We can also imagine that for the film to adopt a barrel shape there need to be the application of an even greater pressure:



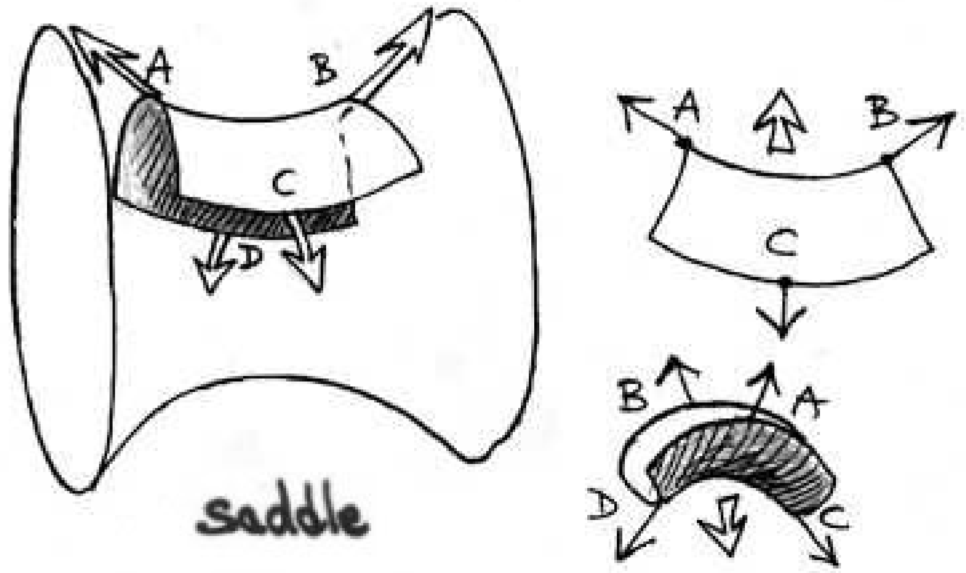
Instead of a film resting on rigid contours we imagine a simple bubble, a surface without edges, an element of this could be visualized as a cap on a sphere:



Here too there is a necessity for a pressure difference. However we could imagine a flat film resting on a circle simply because the result of the pressure forces applied to an element is nil:



But let us return to our film resting on the two coaxial circles where we can clearly see that only a "saddle" shape will allow its existence if there is no pressure difference between the two sides of the circle.



Everyone agreed.

- Boissinière, said Turyshev, you should have gone to art school!

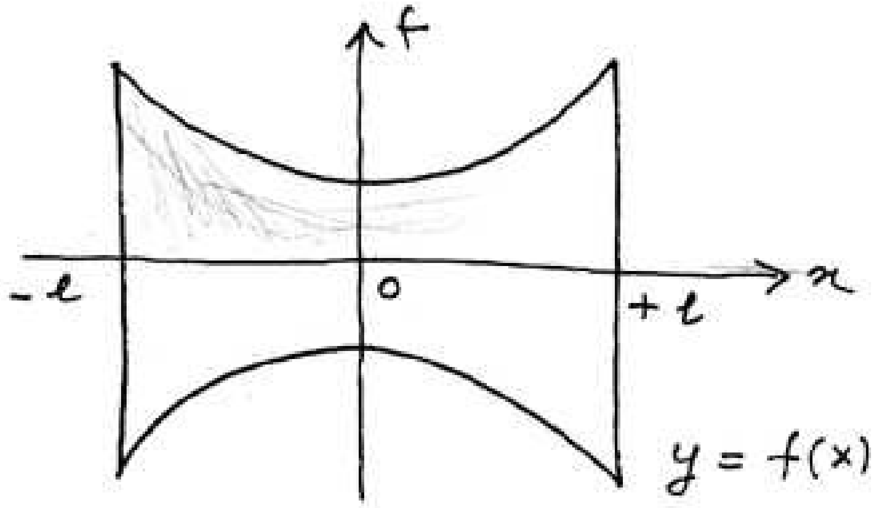
- Thanks, I do my best. That being said, I would like someone to take notes. Could someone look after this?

- No problem, said Fowler who had put his portable computer on the desk in front of him.

- What do you intend to do with your machine?

- I'm used to it. As for the drawings, all I need to do is copy them via this mini-camera then I can join them to the text.

Boissinière continued with Bourbakof's method and tried to give a graphic representation of everything he explained verbally. He drew the following picture:



- I call f (x) the meridian of this surface of revolution. O is the centre of symmetry of the object. My coaxial radius R is set on the abscissas + L and - L.

He turned towards Fowler:

- Can you follow all this?

- Yes.

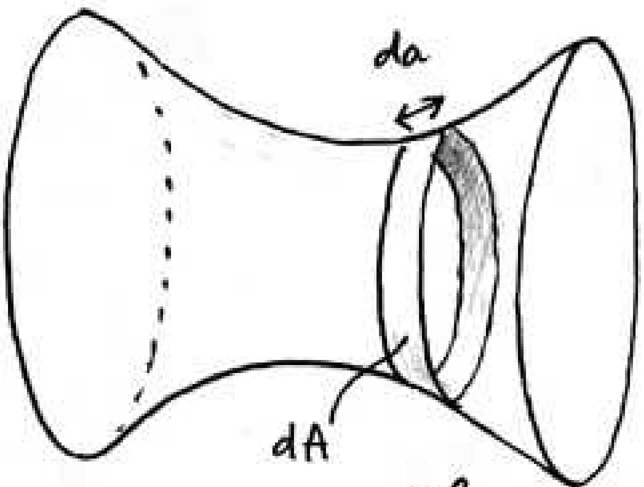
Boissinière approached to have a look.

- But you have put a capital L? ...

- Yes because a small "l" is too similar to the number 1. When we reread all this we have to remember that in the text our capital L corresponds to the small l of the diagram.

Boissinière continued:

- So I can calculate the area element of this surface:



$$da^2 = dx^2 + f'^2 dx^2$$

$$da = \sqrt{1+f'^2}\,dx$$

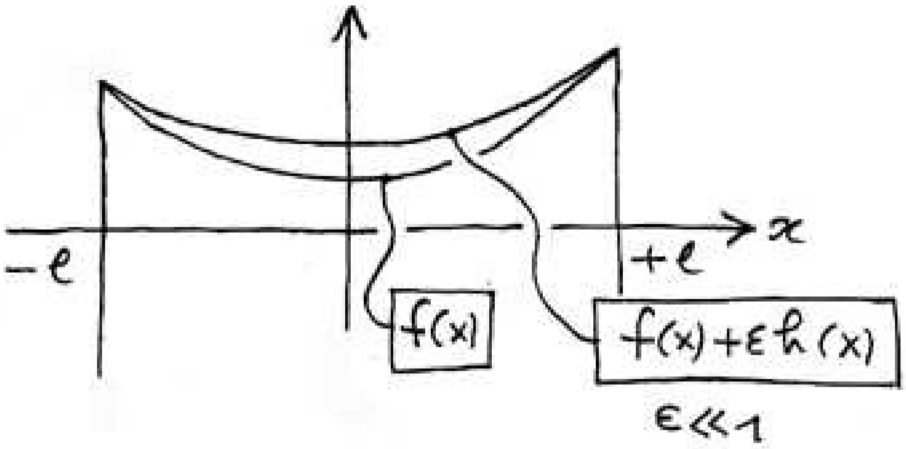
$$dA = 2\pi f\sqrt{1+f'^2}\,dx$$

$$A(f) = \int_{-l}^{+l} 2\pi f\sqrt{1+f'^2}\,dx$$

The surface tensions exerted on the film make it take a configuration corresponding to a minimum area.

Boissinière had lost the thread a little. Turyshev, a biologist spoke up:

- And there Bourbakof has disturbed this surface a bit, and in a way that allows it to keep symmetry of revolution and still rest on the two circles. He said that this came down to adding a term of disturbance h(x) to the function f(x) which represents the meridian:



- OK, OK, I follow you, said Boissinière. Now we calculate the new area based on this meridian, a neighbour of the preceding one:

$$A(f+\varepsilon h)=\int_{-\ell}^{+\ell} 2\pi\,(f+\varepsilon h)\sqrt{1+(f'+\varepsilon h')^2}\;dx$$

- And there you are making a development in series while just keeping the terms of the first order.

Boissinière was having great fun. He could imagine himself twenty years before in some preparatory course or other. He felt he should show all the details of his calculation.

$$A(f+\varepsilon h) = 2\pi \int_{-\ell}^{+\ell} (f+\varepsilon h)\sqrt{1+f'^2+2f'h'\varepsilon+\cancel{\varepsilon^2 h'^2}}\, dx$$

$$= 2\pi \int_{-\ell}^{+\ell} (f+\varepsilon h)\sqrt{1+f'^2}\sqrt{1+\frac{2f'h'\varepsilon}{1+f'^2}}\, dx$$

$$= 2\pi \int_{-\ell}^{+\ell} (f+\varepsilon h)\sqrt{1+f'^2}\left(1+\frac{f'h'\varepsilon}{1+f'^2}\right) dx$$

$$= 2\pi \int_{-\ell}^{+\ell} (f+\varepsilon h)\left[\sqrt{1+f'^2}+\frac{f'h'\varepsilon}{\sqrt{1+f'^2}}\right] dx$$

$$= 2\pi \int_{-\ell}^{+\ell} f\sqrt{1+f'^2}\, dx + 2\pi \int_{-\ell}^{+\ell} \left\{\varepsilon h\sqrt{1+f'^2}+\frac{\varepsilon f f'h'}{\sqrt{1+f'^2}}+\cancel{\frac{\varepsilon^2 h f'h'}{\sqrt{1+f'^2}}}\right\} dx$$

- Which will give us :

$$\lim_{\varepsilon\to 0}\frac{A(f+\varepsilon h)-A(f)}{\varepsilon}=2\pi\int_{-\ell}^{+\ell}\left(h\sqrt{1+f'^2}+\frac{ff'h'}{\sqrt{1+f'^2}}\right)dx$$

$$I_1=2\pi\int_{-\ell}^{+\ell}h\sqrt{1+f'^2}\,dx \qquad I_2=2\pi\int_{-\ell}^{+\ell}\frac{ff'h'}{\sqrt{1+f'^2}}\,dx$$

$h'dx=dh$ $\qquad I_2=2\pi\displaystyle\int_{-\ell}^{+\ell}\frac{ff'dh}{\sqrt{1+f'^2}}$

integration in parts:

$$\int u\,dv=[uv]-\int v\,du$$

$$I_2=2\pi\left[\frac{ff'h}{\sqrt{1+f'^2}}\right]_{-\ell}^{+\ell}-2\pi\int_{-\ell}^{+\ell}h\,d\left(\frac{ff'}{\sqrt{1+f'^2}}\right)dx$$

$\downarrow$

nil because $h(\ell)=h(-\ell)=0$

$$d\left(\frac{ff'}{\sqrt{1+f'^2}}\right) = \frac{(f'^2+ff'')\sqrt{1+f'^2} - ff'\frac{2f'f''}{2\sqrt{1+f'^2}}}{1+f'^2}$$

$$= \frac{(f'^2+ff'')(1+f'^2) - ff'^2f''}{(1+f'^2)^{3/2}} = \frac{f'^2+ff''+\cancel{ff'^2f''}-\cancel{ff'^2f''}+f'^4}{(1+f'^2)^{3/2}}$$

$$\frac{A(f+\varepsilon h)-A(f)}{\varepsilon} = I_1+I_2 = 2\pi\int_{-\ell}^{+\ell} h\sqrt{1+f'^2}\,dx - 2\pi\int_{-\ell}^{+\ell} h\,\frac{f'^4+ff''+f'^2}{(1+f'^2)^{3/2}}\,dx$$

$$\frac{A(f+\varepsilon h)-A(f)}{\varepsilon} = 2\pi\int_{-\ell}^{+\ell} \frac{h\left[1+f'^2-ff''\right]}{(1+f'^2)^{3/2}}\,dx$$

Bourbakof was pleased.

- OK, I see that you haven't lost your touch. Now what does it need for the area variation to become extremal?

Fowler scratched his head.

- It seems to me that if the numerator was nil, in the integral, it should work. As we have developed in series, that means that the variation will be of a superior order so we are definitely in an extremal configuration.

$$1+f'^2-ff''=0$$

Boissinière continued:

- So here there is a differential equation which gives us the corresponding meridian equation for an extremal membrane surface. Is it easy to resolve?

- This is student stuff in a way, said Bourbakof, and I must admit that it has little interest. I'll give you the result :

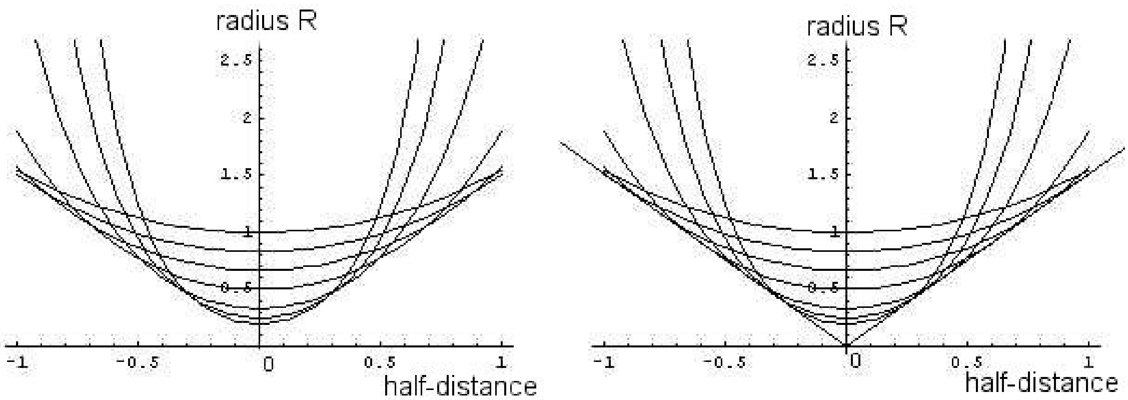
$$\boxed{f(x) = \frac{\mathrm{ch}\, sx}{s}}$$

$$f' = \mathrm{sh}\, sx \quad f'' = s\,\mathrm{ch}\, sx \quad f f'' = \mathrm{ch}^2 sx$$

$$1 + \mathrm{sh}^2 sx - \mathrm{ch}^2 sx = 0 \quad \text{car} \quad \mathrm{ch}^2 sx - \mathrm{sh}^2 sx = 1$$

Fowler remained at his portable.

In an instant he had produced the general look of the curves.



- The envelope for this family of curves is apparently made of two straight lines going through the origin.

- When x is nil the hyperbolic cosine equals the unit. The ordinate of the minimum curve equals 1/s

Boissinière looked at the diagram with attention.

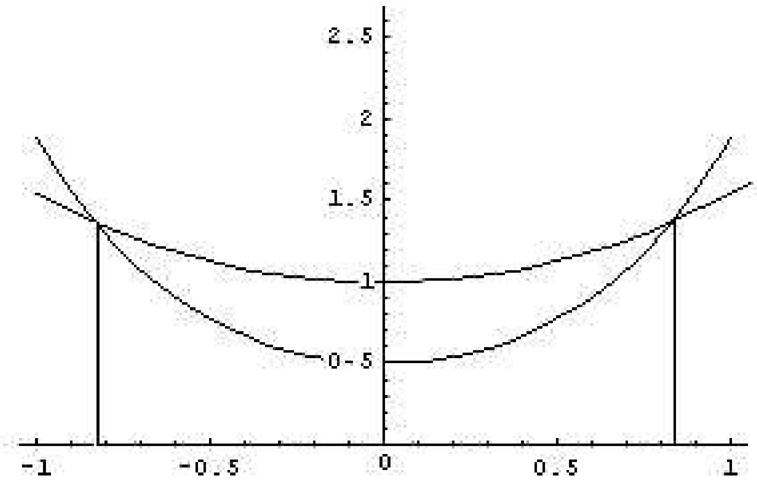
- Something tells me that these curves are ... homothetic.

- Well spotted, said Bourbakof. You have an artist's eye. The equation could be written thus:

$$sf(x) = ch(sx)$$

We can see that if f (x) is a solution, then λ f (λ x) is too.

Turyshev noticed that the two surfaces corresponding to the meridians passed through the two circles:



- Good, said Fowler, without mentioning his remark, now that we have the equation for this meridian we only need to adjust the value of the parameter s in such a way as to get this surface to rest on the two coaxial circles of radius R, situated at + L and - L.

- Yes but we are going to see that that is not always possible, Bourbakof remarked with a wry smile.

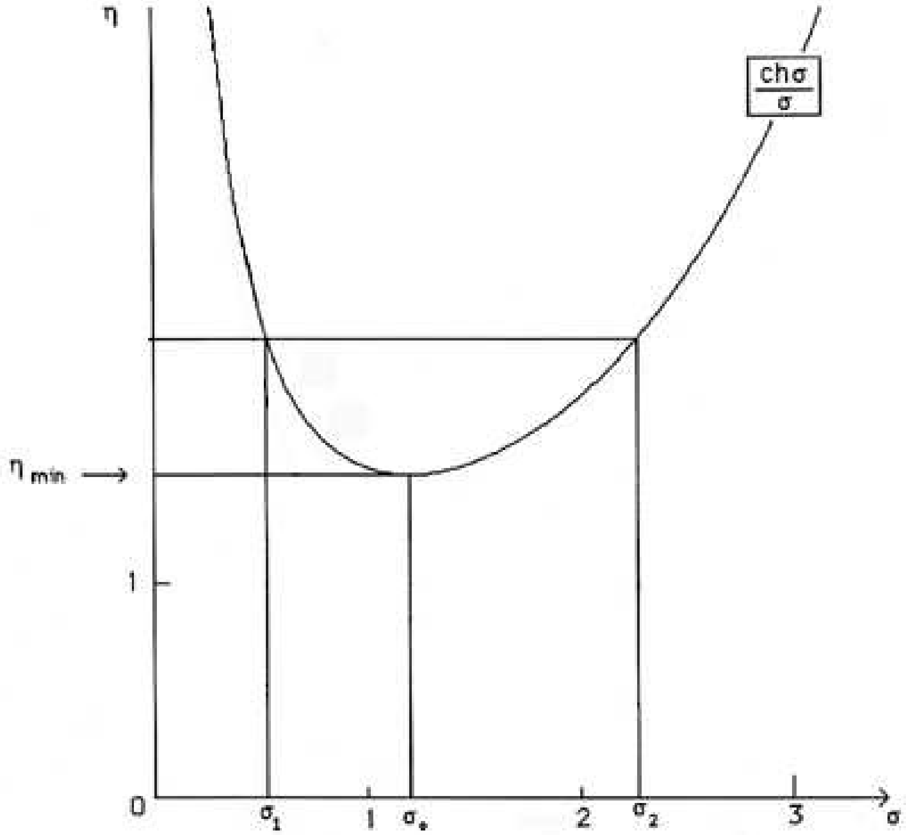
- OK... I can always write:

$$f s = ch\, s x$$

$$R s = ch\, s \ell \qquad \text{on pose : } \sigma = s \ell$$

$$\sigma \frac{R}{\ell} = ch\, \sigma \qquad \frac{R}{\ell} = \frac{ch\, \sigma}{\sigma} \qquad \text{on pose } \eta = \frac{R}{\ell}$$

There is a curve to draw. Ah, effectively!



For η = 2 we have two values.

- You see, exclaimed Turyshev, that corresponds to the solution with two meridians that I set out above. If we take η for example, that is to say a ratio between the radius R of the circles and the half-distance L separating them, equal to 2, what will be the two values of σ?

Fowler tapped on his keyboard.

- Give me five minutes and I'll calculate the numeric values. There we are..:

$\sigma_1 = 0{,}5894$

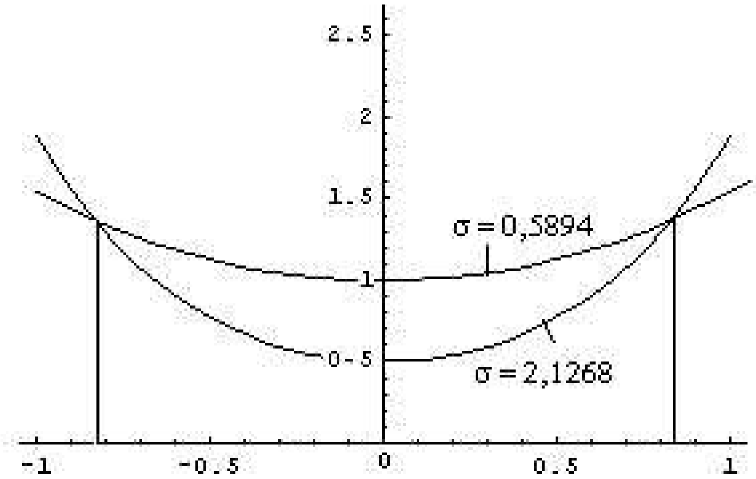
$\sigma_2 = 2{,}1268$

So we have the two meridian solutions to the differential equations. Let us see how we can vary the radius of the circle of the neck of the meridian, that is to say the circle with the minimum radius which is at x = 0:

$$f = r = \frac{\mathrm{ch}\, s x}{s} = \frac{\mathrm{ch}\left(\sigma \frac{x}{L}\right)}{\sigma} L$$

$$x = 0 \longrightarrow r_{min} = \frac{L}{\sigma}$$

If we get two different meridians for the same value of L and R, the smallest radius (neck circle) will be obtained for the highest value of σ (that is 2, 1268).



The smallest area corresponds to the smallest s so to the largest neck circle. But which of these two surfaces of revolution has the smallest surface?

Bourbakof recommended this calculation :

$$f = \frac{\operatorname{ch} sx}{s} \qquad f' = \operatorname{sh} sx$$

$$A(f) = \int_{-L}^{+L} 2\pi f \sqrt{1 + f'^2}\, dx = \int_{-L}^{+L} \frac{2\pi}{s} \operatorname{ch}^2 sx\, dx$$

$$sx = u \qquad dx = \frac{du}{s}$$

$$A(f) = \int_{-sL}^{sL} \frac{2\pi}{s^2} \operatorname{ch}^2 u\, du \qquad \operatorname{ch} u = \frac{e^u + e^{-u}}{2}$$

$$\operatorname{ch}^2 u = \frac{e^{2u} + e^{-2u} + 2}{4} = \frac{1}{2}\left(\operatorname{ch} 2u + 1\right)$$

$$A(f) = \int_{-sL}^{+sL} \frac{\pi}{s^2}\left(\operatorname{ch} 2u + 1\right) du = \frac{\pi}{s^2}\left[\frac{1}{2} \operatorname{sh} 2u + u\right]_{-sL}^{+sL}$$

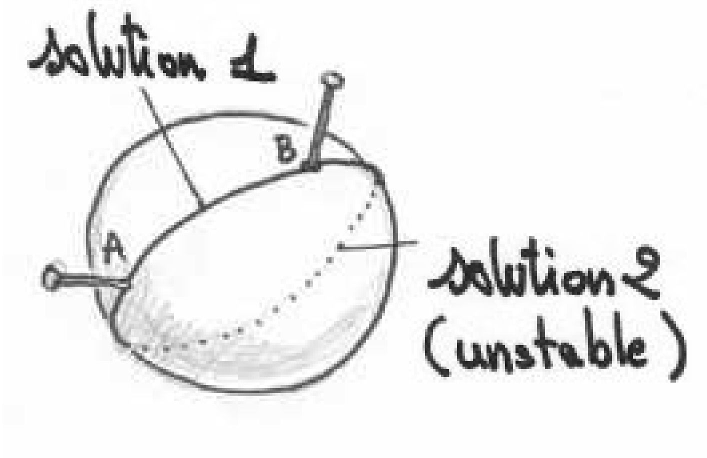
$$A(f) = \frac{\pi}{s^2}\left(\operatorname{sh} 2sL + 2sL\right)$$

on fait $L = 1$ $\qquad A \sim \dfrac{\operatorname{sh} 2s + 2s}{s^2}$

- Fowler, calculates that for us.

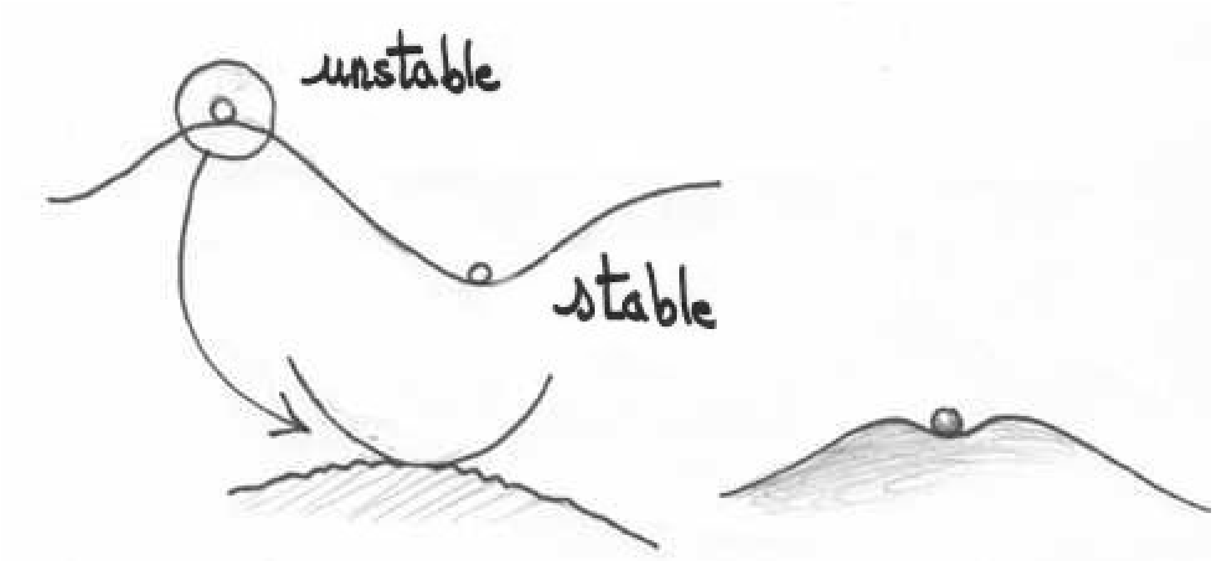
- It's done: the meridian having the largest neck radius is the one with the smallest area. However I have trouble in imagining that there can be two solutions. Please. Bourbakof, get us out of this impasse.

- One of the solutions is simply unstable. You need to effect a series development to the order of two to show up this instability. That will be a little complicated. Let us take a clearer example. Imagine that you are looking for the minimum trajectory between two points on a sphere. We could treat that in an analogous manner, like a problem of extremals. In this case we would find two solutions :



- It looks like an elastic joining two pins stuck in an orange.

- There are two trajectories AB therefore. Both are extremal trajectories but only one is "stable", authentically shorter. That is the difference between mathematics and physics. A physics solution is a stable solution, as the following figure shows:



The ball at the top of the hill on the left is in unstable equilibrium.

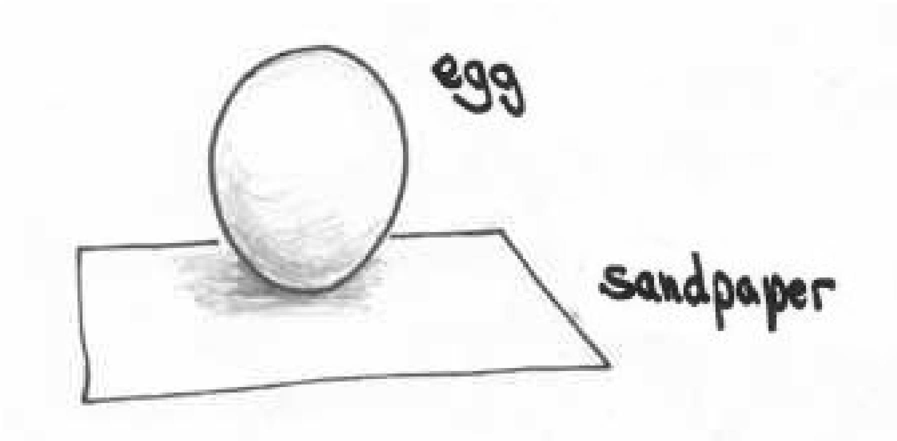
- Is there no way in which we can manage to put it exactly at the summit?

- No, if you managed to keep it at the summit of the hill then the hill could not be perfectly smooth. The least roughness distorts the problem; it is like putting the ball in a small dip.

- Hmm, said Iliouchine, under those conditions we should be able to balance an egg on its point providing that it is set on large grained sandpapers. Boissinière, do you have any eggs and sandpaper?

- Yes, but not in the same place. I'll get you all that.

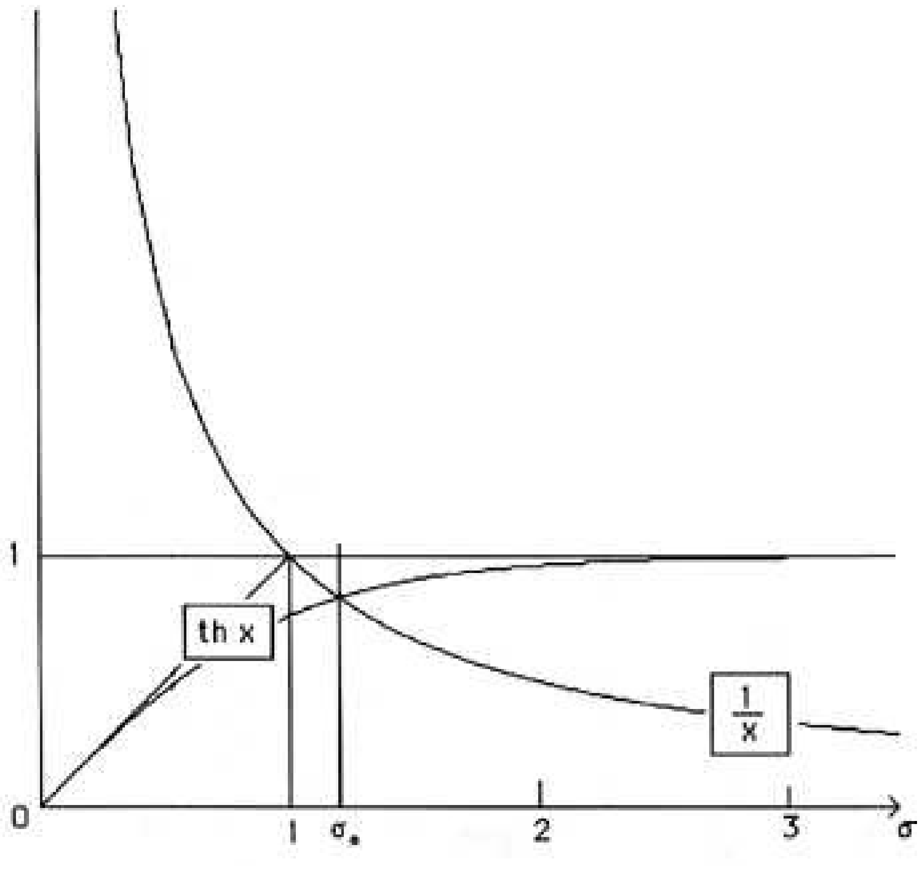
The experiment was a success.



- Let us return to the figure which gave us two values of σ. You can see that a horizontal straight line doesn't automatically cut this curve. For low values of η, there is no solution. There is a critical value, close to $\eta_{cr} = 1{,}5$. To find the corresponding value of σ it is simply a matter of derivation:

$$\left(\frac{ch\,\sigma}{\sigma}\right)' = \frac{\sigma\, sh\,\sigma - ch\,\sigma}{\sigma^2} \quad \textbf{nil for} \quad th\,\sigma_0 = \frac{1}{\sigma_0}$$

With the following graphic representation:



That gives us a value for σ close to 1,2. It just remains to inject this value to find the maximum distance over which we can separate the two coaxial circles of R radius.

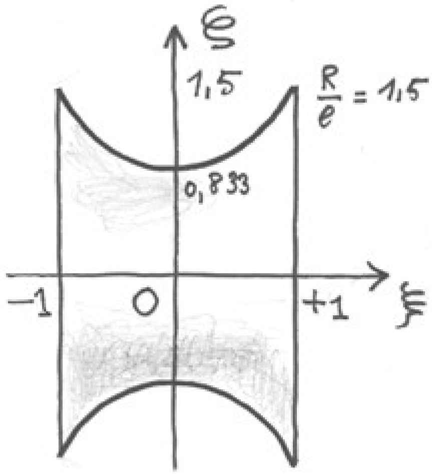
Let $\zeta = \frac{x}{\ell}$

with $\ell = 1$

$$s = \ell\sigma = 1{,}2$$

$$\zeta = \frac{ch\, 1{,}2x}{1{,}2}$$

$$\zeta_{min} = \frac{1}{1{,}2} = 0{,}833$$

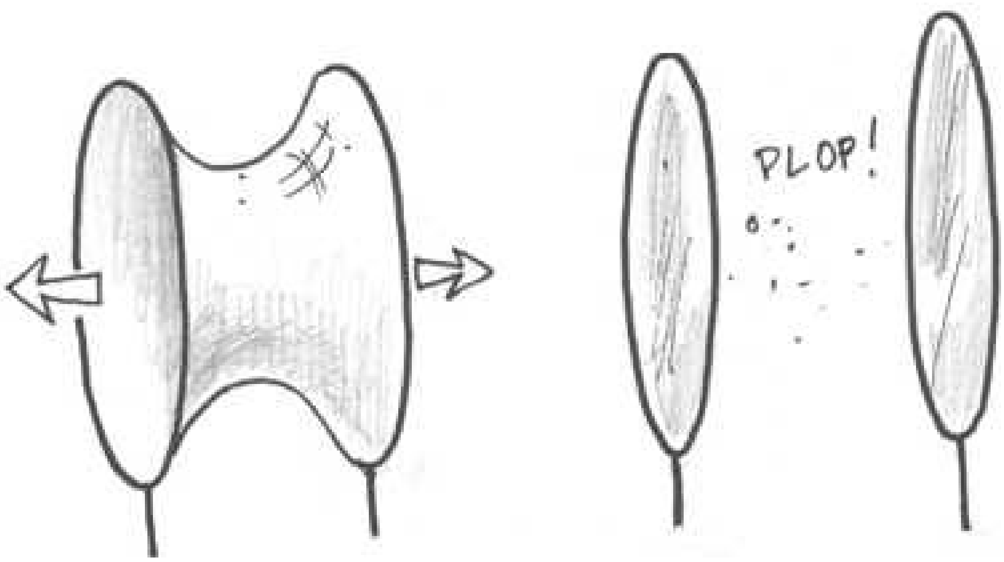


Iliouchine had arrived late and hadn't been able to do the experiment with the soapy water.

- What happens when we separate the two circles further?

- Try it my friend!

In a second he was seated at the table opposite the accessories and a bowl filled with soapy water. After a few tries he managed to get the required configuration and he observed this:



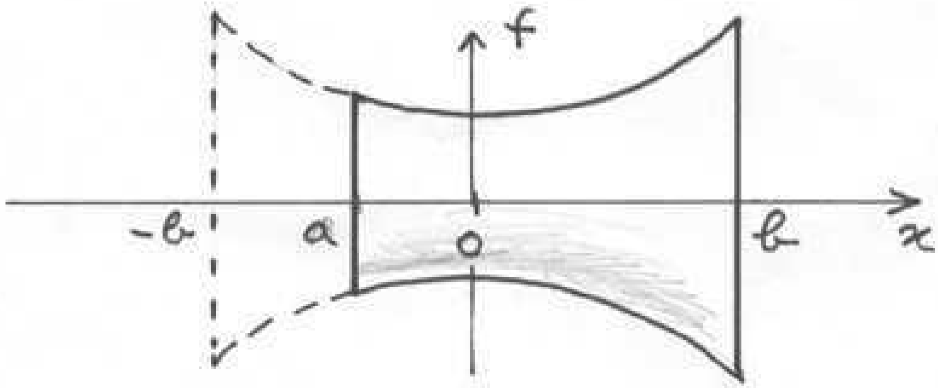
- Very amusing! The membrane broke and made two flat disks.

- Still with a configuration of minimal surface area. However this surface is made up of two unconnected disks.

Boissinière was having great fun.

- Bourbakof, how would we treat the problem if it concerned circles of different radii?

- Let us look at the calculation. There I took two circles with the same radius to help us understand, but that isn't obligatory. In the end you will have the same differential and the same solution with its two integration constants. The minimum surface rating on circles with different radii will simply be a part of the type of surface that we found earlier. By taking two circles of the same value we simply simplified things to show the critical separation conditions.



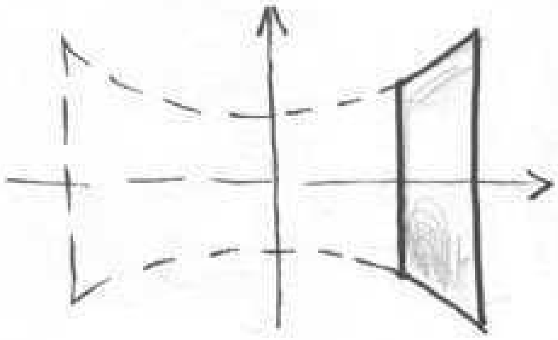
$$A(f) = \int_a^b 2\pi f \sqrt{1+f'^2}\, dx$$

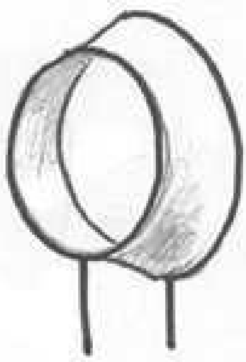
$$A(f+\varepsilon h) = \int_a^b 2\pi (f+\varepsilon h)\sqrt{1+(f'+\varepsilon h')^2}\, dx$$

$$\downarrow$$

$$\underset{\varepsilon\to 0}{\frac{A(f+\varepsilon h)-A(f)}{\varepsilon}} = 2\pi\int_a^b \frac{h\left[1+f'^2-ff''\right]}{(1+f'^2)^{3/2}}\, dx$$

$$1+f'^2-ff''=0 \longrightarrow f(x)=\frac{\operatorname{ch} s(x-x_0)}{s}$$





- Excuse me, it was obvious...

Fowler was pleased.

- Do you see Iliouchine, Bourbakof has shown a very pretty variational calculation for a basin filled with soapy water. You are a magician my friend.

Bourbakof was insensitive to the praise.

- It is also a way to get a Lagrangian.

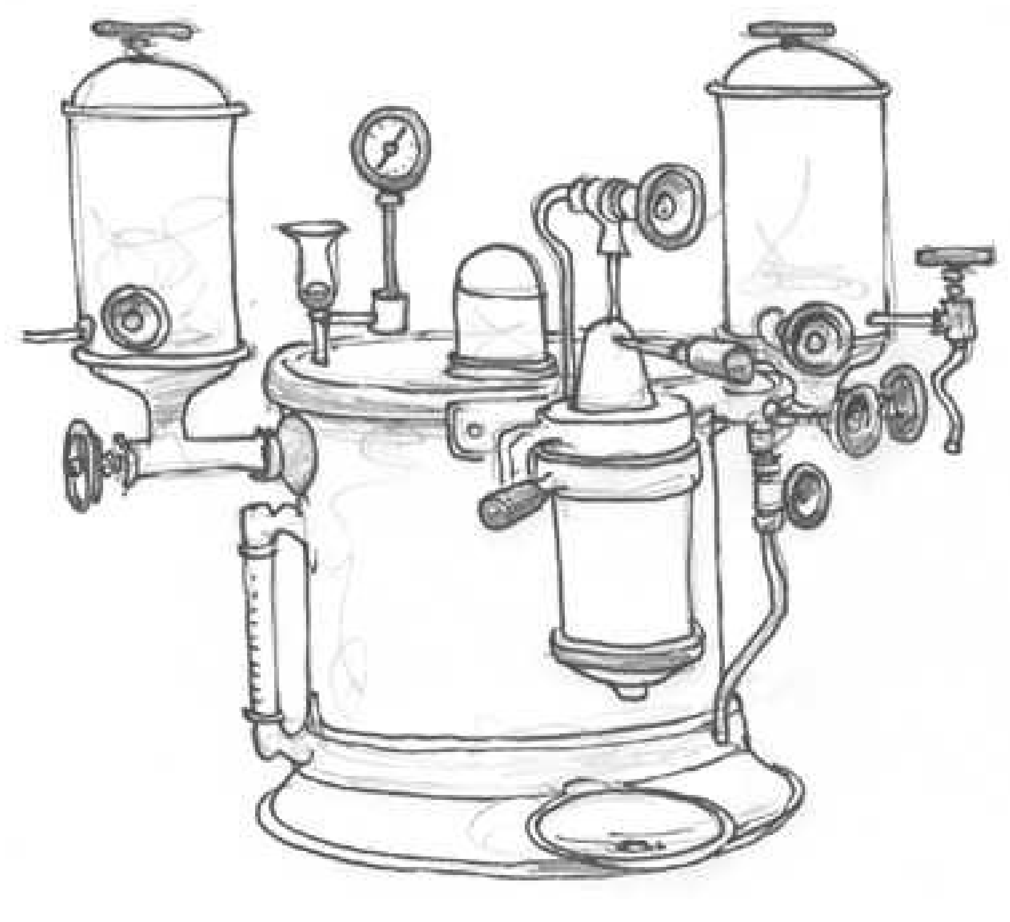
- A Lagrangian, exclaimed Iliouchine! I've never seen one in my life, despite the fact that my theoretical physics colleagues were always talking about them.

- You now have one in front of you, Fowler remarked.

- Where? Iliouchine asked, his eyes wide open as they ran over the calculations on the blackboard.

Boissinière burst out laughing.

- I suggest that we go for a coffee then, under Bourbakof's guidance, we can initiate our biologist friend Turyshev into the secrets of the Lagrangian.



- Good Lord! exclaimed Fowler, Boissinière, where did this monstrosity come from?

- From Czechoslovakia. But I can assure you that it makes excellent coffee. Turyshev, who has analysed it, speaks highly of its quality.

- I wouldn't automatically believe everything he says Boissinière. In fact this percolator comes directly from the wreck of the Titanic!

- That doesn't surprise me one bit!

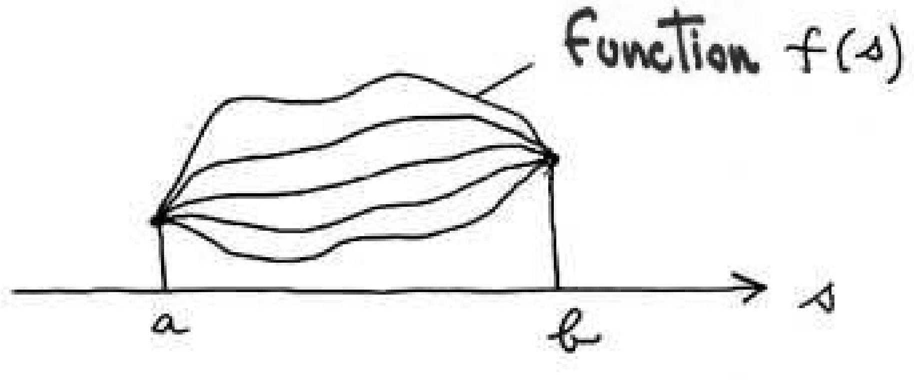
Once the coffee had been drunk everyone returned to what had now become the seminar room. Turyshev was sent to the blackboard where Bourbakof addressed him:

- Turyshev, would you accept a slight change in notation? Let us replace the x variable with s and choose to mark the derivation in relation to this variable with a point placed above it. Then we will consider a function L which depends on the function f and its derivation. We will call this function a ***LAGRANGIAN***.

Guided by Bourbakof, Turyshev did so.

$$\mathcal{L} \equiv \mathcal{L}\,(f, \dot{f}) \text{ where } \dot{f} = \frac{df}{ds}$$

- OK. The function f (s) will be designed with a certain interval $[a,b]$. We will consider the ensemble of functions (s) that take the values given in s = a and in s = b:



We will then include the function

$$L(f(s), \dot{f}(s))$$

on the interval (a, b) and we'll call this integral an action integral or, more simply, an ***ACTION***.

$$A(f) = \int_a^b \mathcal{L}\,(f, \dot{f})\,ds \quad \text{(action)}$$

Now let us return to the question asked by Lagrange two hundred years ago:

"Among these functions f (s) do any verify:

$$f(a) = \alpha \qquad et \qquad f(b) = \beta$$

a particular function which makes this action extremal? »

Turyshev, go on.

The biologist was disconcerted for a moment.

- I suppose, Turyshev ... that this must be similar to what we did earlier.
- Yes, exactly.
- OK, I shall introduce a slight perturbation in the function f (s):

$$f(s) \longrightarrow f(s) + \varepsilon h(s) \qquad \varepsilon \ll 1$$

With:

$$h(a) = h(b) = 0$$

so that each of the perturbed functions takes the same values as f at the endpoints a and b.

To solve this problem of extremum I will consider the passage to the limit of the ratio:

$$\lim_{\varepsilon \to 0} \frac{A(f + \varepsilon h) - A(f)}{\varepsilon}$$

I will effect a development of my first order integral.

$$A(f + \varepsilon h) = \int_a^b \mathcal{L}(f, \dot{f}) ds$$

$$+ \varepsilon \int_a^b \left[\frac{\partial \mathcal{L}}{\partial f} h + \frac{\partial \mathcal{L}}{\partial \dot{f}} \dot{h} \right] ds \quad + o(\varepsilon)$$

This allows me to write:

$$\lim_{\varepsilon \to 0} \frac{A(f+\varepsilon h) - A(f)}{\varepsilon} = \int_a^b \left[\frac{\partial \mathcal{L}}{\partial f} h + \frac{\partial \mathcal{L}}{\partial \dot{f}} \dot{h} \right] ds$$

there I have two integrals:

$$I_1 = \int_a^b \frac{\partial \mathcal{L}}{\partial f} h \, ds \; ; \; I_2 = \int_a^b \frac{\partial \mathcal{L}}{\partial \dot{f}} \dot{h} \, ds$$

In the second I will negotiate an integral in parts, by analogy with what we did earlier.

- OK...

- I just need to write:

$$\dot{h} \, ds = dh \qquad\qquad I_2 = \int_a^b \frac{\partial \mathcal{L}}{\partial \dot{f}} \, dh$$

and off we go for this integration in parts with the same trick as we used earlier:

$$I_2 = \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \dot{f}} h \right]_a^b}_{\text{nil}} - \int_a^b h \left[\frac{d}{ds} \left(\frac{\partial \mathcal{L}}{\partial \dot{f}} \right) \right] ds$$

because $h(a) = h(b) = 0$

Which comes from the fact that all the functions f (s) which are in the Lagrangian, in the same way as their derivations only differ from each other in the function h (s), such as h(a) =0 and h(b) =0.

So this remains :

$$\lim_{\varepsilon \to 0} \frac{A(f+\varepsilon h) - A(f)}{\varepsilon} =$$

$$\int_a^b h\left[\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right)\right] ds$$

The ratio will be nil if the quantity between square brackets is nil.

- Which gives you Lagrange's equation, in "one-dimensional" form:

Turyshev did it :

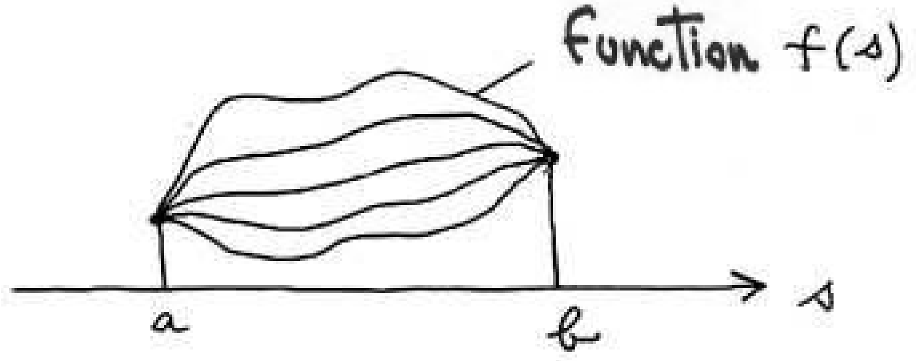
$$\boxed{\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right) = \frac{\partial \mathcal{L}}{\partial f}}$$

Lagrange

You see my dear Turyshev that, in the end, it is no big deal.

- It's astonishing. I've always looked at this equation with the greatest perplexity.

- Let us go back to the figure:



You can see that this equation appears as soon as we try to clear the "path" where a certain "action" is extremized. We are looking for a particular path among all of the paths that pass through two given points.

- What is strange is that it comes up in this minimal surface area calculation.

- Look at your notes. By replacing the "prime" of the derivation with a point, what is your action integral?

- Hmm, I would say that it is:

$$A(f) = \int_a^b 2\pi f \sqrt{1+\dot{f}^2}\, dx$$

- And the Lagrangian ?

- Humm...

$$\mathcal{L} \equiv \mathcal{L}(f, \dot{f}) = 2\pi\sqrt{1+\dot{f}^2}$$

- And if you write the Lagrange equation starting with the Lagrangian what do you get?

Turyshev seemed to be in a dream.

- I'm going to remove the 2π so that I don't have to carry it around.

$$\mathcal{L} \equiv \mathcal{L}(f,\dot{f}) = f\sqrt{1+\dot{f}^2}$$

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right) = \frac{\partial \mathcal{L}}{\partial f} = \sqrt{1+\dot{f}^2} = \frac{(1+\dot{f}^2)^2}{(1+\dot{f}^2)^{3/2}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{f}} = \frac{f\dot{f}}{\sqrt{1+\dot{f}^2}}$$

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right) = \frac{(\dot{f}^2 + f\ddot{f})\sqrt{1+\dot{f}^2} - f\dot{f}\frac{\dot{f}\ddot{f}}{\sqrt{1+\dot{f}^2}}}{(1+\dot{f}^2)}$$

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right) = \frac{(\dot{f}^2 + f\ddot{f})(1+\dot{f}^2) - f\dot{f}^2\ddot{f}}{(1+\dot{f}^2)^{3/2}}$$

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{f}}\right) = \frac{\dot{f}^2 + f\ddot{f} + \dot{f}^4 + f\cancel{\dot{f}^2}\ddot{f} - \cancel{f\dot{f}^2}f''}{(1+\dot{f}^2)^{3/2}}$$

$$\frac{\dot{f}^2 + f\ddot{f} + \cancel{\dot{f}^4}}{(1+\dot{f}^2)^{3/2}} = \frac{\cancel{\dot{f}^4} + 2\dot{f}^2 + 1}{(1+\dot{f}^2)^{3/2}}$$

$$1 + \dot{f}^2 - f\ddot{f} = 0$$

And I find the differential equation of the extremal surface meridian.

There was applause in the room. Fowler whistled through his fingers.

- Do you know Turyshev, you're very good for a biologist. I could easily take you for a thesis student.

- Don't exaggerate. But it's true that I have learnt a lot today.

Turyshev smiled. Boissinière continued:

- Clearly you are an artist Bourbakof. You have just shown us the Lagrange equation for a soap bubble. You are someone precious to have on board my friend.

Bourbakof tried to look modest.

- In the world of mathematics making an effort to be understandable can almost be considered treason.

- In that case you are a successful traitor.

- While we are on the subject, why not settle this business of the Lagrange equation in a variable number of dimensions?

Turyshev looked astonished:

- Because Lagrange's equation also works in a universe with several dimensions?

- But of course my friend; thirty two if you fancy. Don't sit down so fast. Do you feel up to continuing?

- Err... yes.

Bourbakof looked at his watch.

- After we'll eat, I promise. But while the iron is hot, let us strike it.

- OK, the biologist replied.

Everyone went back to their place. Boissinière questioned Bourbakof:

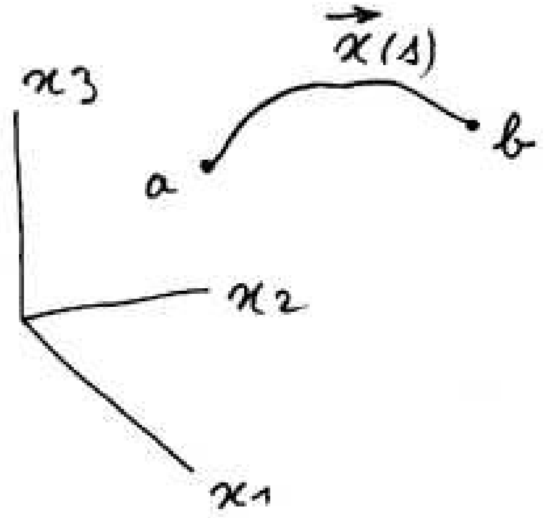
- Are you sure that this isn't too much in one go for a biologist?

It was Turyshev who reassured the audience:

- It is fairly new for me but I assure you I am having lots of fun.

- OK, Bourbakof continued, Lagrange's equation can be built taking any number of dimensions $x_1, x_2, \cdots, x_n$. It will be more practical to imagine a three dimensional space x_1, x_2, x_3 that you will obviously identify with the familiar three dimensional Euclidean, though it will be better to limit ourselves to calling it "$\mathbb{R}^3$ space". Still never mind. In this three dimensional space let us imagine two points a and b and a path joining these two points, decided by one of the parameters. The coordinates $\left(x_1, x_2, x_3\right)$ of the point are functions $f^i\left(s\right)$. Turyshev, draw us a 3D space and a trajectory joining any two points a and b.

Turyshev did so.



$$\vec{x}(s)\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ x_3 = f_3(s) \end{cases}$$

$$\vec{f}\begin{cases} f_1(s) \\ f_2(s) \\ f_3(s) \end{cases}$$

- OK, now derive the functions in relation to the parameter s.

Turyshev wrote :

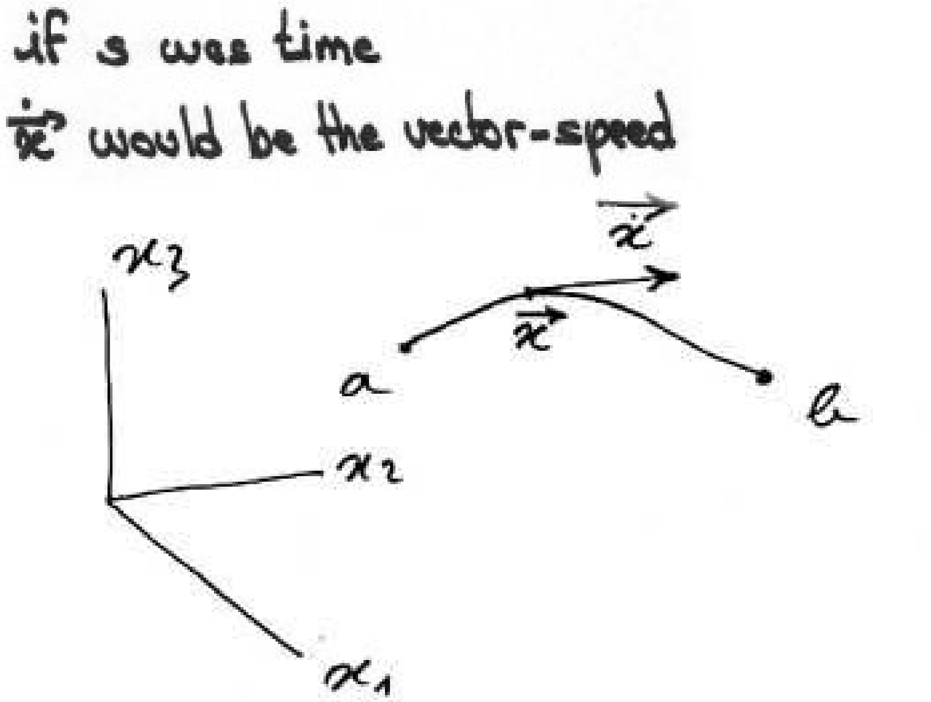
$$\dot{\vec{x}}\begin{cases} \dot{x}_1 = \dot{f}_1 = \dfrac{dx_1}{ds} \\ \dot{x}_2 = \dot{f}_2 = \dfrac{dx_2}{ds} \\ \dot{x}_3 = \dot{f}_3 = \dfrac{dx_3}{ds} \end{cases}$$

Bourbakof got up, took the chalk and wrote:

- We could just write :

$$\vec{\dot{x}} \begin{cases} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{cases}$$

At this point the physician could have asked himself what we were playing at. To reply "analytical mechanics" might not have been of much use to him. But :



This said, the stories of Lagrangians and extremums are not just a simple problem of kinematics in a 3D Euclidean space. Just a remark to give you a mental image. I get the feeling that you suspect me of trying to make you lose contact with reality.

- Me? I never said anything, Turyshev exclaimed.

- You seem reticent. Is it just an impression I have?

- I assure you I feel fine, carry on...

- So we can give ourselves a Lagrangian, which is always a scalar. A short while ago this had been defined from a function s, which was also a scalar. Now we only need to imagine that this function **f** (or **x**) is a "sort of vector" with "components" f^i. One could then define our Lagrangian from this vectorial function and its derivation.

Turyshev wrote :

$$\mathcal{L}(\vec{f},\dot{\vec{f}}) = \mathcal{L}(f_1,f_2,f_3,\dot{f}_1,\dot{f}_2,\dot{f}_3)$$

- Very good! Now replace these functions f_i and their derivations by the functions xi and their derivations, using the same notation.

- At your orders professor

$$\mathcal{L} \equiv \mathcal{L}(\vec{x},\dot{\vec{x}}) = \mathcal{L}(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3)$$

- As before, we are going to define an ***action.***

- It's a scalar?

- Yes, it's always a scalar:

$$\text{Action}: \; A(\vec{x}) = \int_a^b \mathcal{L}(\vec{x},\dot{\vec{x}})\,ds$$

It's an "***integral of action***" that is calculated along a trajectory joining two points a and b of this 3D space, an integral that is defined by the datum of this function that we call a Lagrangian. We will look for the trajectory that makes this action extremal. Get your inspiration from what we did earlier...

Turyshev felt inspired.

$$\begin{cases} \text{we form } \mathcal{L}(\vec{x}+\varepsilon\vec{h},\ \dot{\vec{x}}+\varepsilon\dot{\vec{h}}) \\ (\text{ or } \mathcal{L}(\vec{f}+\varepsilon\vec{h},\ \dot{\vec{f}}+\varepsilon\dot{\vec{h}}) \end{cases}$$

- You must understand what we are doing, remarked Turyshev. The "arrowed x" represents a trajectory ab, which is the ensemble of functions:

$$x_i(s) = f_i(s)$$

so that for all possible functions, that is to say for all possible paths:

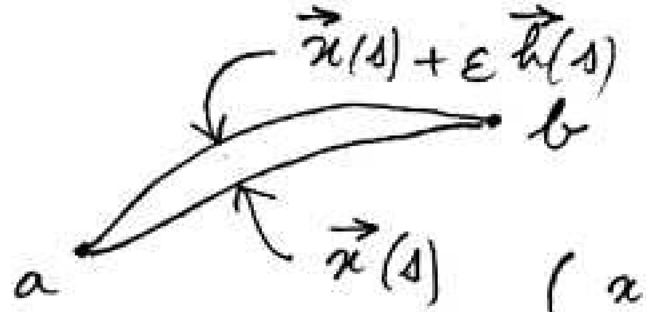
$$f_i(a) = \alpha_i \quad et \quad f_i(b) = \beta_i$$

That is the same as saying that all these paths pass though the points a and b of the space (x_1, x_2, x_3). That also means that:

$$h_i(a) = h_i(b) = 0$$

- OK...

- Let us imagine a second trajectory that is near the first:



$$\begin{cases} x_1(s) + \varepsilon\, h_1(s) \\ x_2(s) + \varepsilon\, h_2(s) \\ x_3(s) + \varepsilon\, h_3(s) \end{cases}$$

The value of the action integral, which is a scalar, will not necessarily be different according to the path taken to effect this integration of s.

$$A(\vec{x}) \neq A(\vec{x} + \varepsilon \vec{h})$$

- Logical.

- By applying this "variation" let us decide to look for the curve or curves and the path or paths that make this action extremal. So lay out the calculation as before.

Turyshev did so.

$$\mathcal{L}(\vec{x}+\varepsilon\vec{h}, \dot{\vec{x}}+\varepsilon\dot{\vec{h}}) = \mathcal{L}(\vec{x}, \dot{\vec{x}})$$

$$+\varepsilon\left(\frac{\partial\mathcal{L}}{\partial\vec{x}}\cdot\vec{h} + \frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}}\cdot\dot{\vec{h}}\right)$$

$$\frac{\partial\mathcal{L}}{\partial\vec{x}}\cdot\vec{h} = \frac{\partial\mathcal{L}}{\partial x_1}h_1 + \frac{\partial\mathcal{L}}{\partial x_2}h_2 + \frac{\partial\mathcal{L}}{\partial x_3}h_3$$

- Very good. You have made a scalar product appear, symbolized by the point.

$$\frac{\partial\mathcal{L}}{\partial\vec{x}}\cdot\vec{h} = \text{"scalar product"}\ \overrightarrow{\left(\frac{\partial\mathcal{L}}{\partial x}\right)}\ \text{with}\ \vec{h}$$

Bourbakof, pleased by the talent of his pupil, made a few comments.

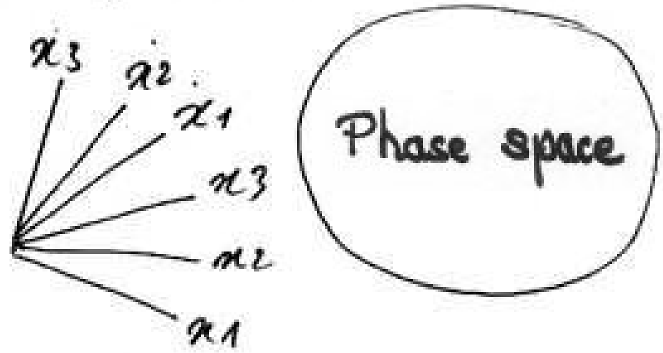
- A mathematician would have written and set out these things differently. But mathematicians and physicians (of the old school) don't use the same language. For example, a physician knows that a function φ, scalar, is defined in a space (x, y, z). It could be a temperature field T(x, y, z) for example. The following expression :

$$\varphi(x, y, z) \qquad \left\{ \begin{array}{l} \dfrac{\partial\varphi}{\partial x} \\[2mm] \dfrac{\partial\varphi}{\partial y} \\[2mm] \dfrac{\partial\varphi}{\partial z} \end{array} \right. \ \text{gradient}$$

would also be familiar to him. He'd call it a "gradient". In fact we can see the appearance of a sort of "six-legged gradient" in our calculation, from the function L of the Lagrangian.

$$\left\{\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial x_1} \\ \dfrac{\partial \mathcal{L}}{\partial x_2} \\ \dfrac{\partial \mathcal{L}}{\partial x_3} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{x}_1} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{x}_2} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{x}_3} \end{array}\right.$$

Sort of "6D gradiant" in space



if $\vec{x}$ = position

if s = time

then $\dot{\vec{x}}$ = speed and

phase space = space

(position, speed) = $(\vec{x}, \dot{\vec{x}})$

A gradient defined in the ***phase space***, such as the function L, like the Lagrangian itself, which is a mathematical being that ***"lives in the phase space".*** As we need to obtain a scalar in the end we will multiply the sort of six-legged gradient below, by another "6-vector". First I will write :

$$\varepsilon \vec{h} \begin{cases} \varepsilon h_1(s) \\ \varepsilon h_2(s) \\ \varepsilon h_3(s) \end{cases}$$

which represents the perturbation of the "vectorial function", which is just a part. This 6-vector is:

$$\begin{cases} \varepsilon h_1 \\ \varepsilon h_2 \\ \varepsilon h_3 \\ \varepsilon \dot{h}_1 \\ \varepsilon \dot{h}_2 \\ \varepsilon \dot{h}_3 \end{cases} \quad \textbf{6-vector } (\varepsilon \vec{h}, \varepsilon \dot{\vec{h}})$$

A simple digression. Turyshev, finish this calculation for us. It is in 3D but could have any number of dimensions.

- OK, said Turyshev, I'll calculate:

$$\lim_{\varepsilon \to 0} \frac{A(\vec{x} + \varepsilon \vec{h}) - A(\vec{x})}{\varepsilon}$$

$$\textbf{but } \dot{h}^i = \frac{dh^i}{ds}$$

$$\textbf{we obtain: } \sum_{i=1}^{n} \int_a^b \left(\frac{\partial \mathcal{L}}{\partial x^i} h^i ds + \frac{\partial \mathcal{L}}{\partial \dot{x}^i} dh^i \right)$$

I'll redo the partial integration trick:

$$\sum_{i=1}^{n} \int_a^b \frac{\partial \mathcal{L}}{\partial \dot{x}^i} dh^i = \sum_{i=1}^{n} \left[h^i \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right]_a^b - \sum_1^n \int_a^b h^i \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} ds$$

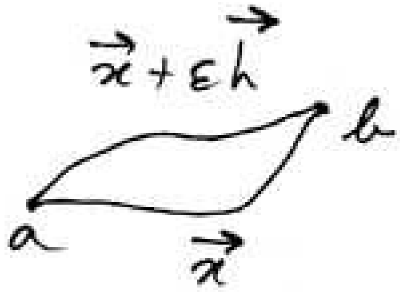
- Very good.

- The thing between square brackets gives zero, function h being nil at the extremities of the path AB

$$\text{point } a \; (s=a) \begin{cases} x1(a) \\ x2(a) \\ x3(a) \end{cases} \qquad \text{point } b \; (s=b) \begin{cases} x_1(b) \\ x_2(b) \\ x3(b) \end{cases}$$

$$h^i(a) = h^i(b) = 0$$

$$\vec{h}(a) = \vec{h}(b) = 0$$



- Exactly.

- It just remains to finalize:

$$\lim_{\varepsilon \to 0} \frac{A(\vec{x} + \varepsilon \vec{h}) - A(\vec{x})}{\varepsilon} \quad \text{tends towards (at 1st order)}$$

$$\sum_{i=1}^{n} \int_a^b h^i \, ds \left[\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right]$$

For the path **x**(s) followed in this space, between the two fixed points a and b which correspond to an extremal of the action A, it is necessary and sufficient that the quantity between the square brackets be nil, which gives us n Lagrange equations :

$$\boxed{\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = \frac{\partial \mathcal{L}}{\partial x^i}}$$ Lagrange

v

Turyshev was thinking:

- I would like to return to what was said earlier concerning the number of dimensions so as to be sure that I have understood.

- Please do so my friend...

- Starting from a space with n dimensions that we will call (x_1, x_2, ..., x_n). This is a vectorial space isn't it?

- Yes.

- So I could symbolize any point in this space with a "vector **x**" by writing:

$$x = (x_1, x_2, \cdots, x_n)$$

In this space I could position two points A and B (which we called a and b above, but it doesn't matter). A curve Γ of this space is an ensemble of functions:

$$\left(x_1(s), x_2(s), \cdots, x_n(s)\right)$$

which depend on a unique parameter s. The Lagrange equations set out above constitute an ensemble of differential equations of the second order. There are n unknown functions to be determined, so I need n equation, and I find that that is what I get from the "Lagrange equation". I can write this series of differential equation thus :

$$\begin{cases} \phi_1(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n) = 0 \\ \phi_2(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n) = 0 \\ - \quad - \quad - \quad - \quad - \\ \phi_n(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n) = 0 \end{cases}$$

Let me remark that in relation to the most general system of equations, s does not explicitly appear.

Bourbakof burst out laughing:

- Turyshev, do you know what you are?

- No...

- You're an upset mathematician.

- Don't joke. I'm always on the edge of a precipice.

- Alright, I'll stop kidding. Carry on.

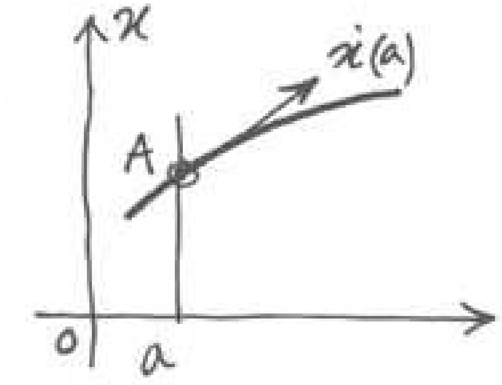
- What bothers me is that, classically, I learnt that solutions for such systems depended on "initial conditions". If I take a differential equation of the second order, making reference to a function x(s), I will have:

$$\phi(x, \dot{x}, \ddot{x}) = 0$$

If I decide to start from a point A such as s = a then my initial conditions will be:

$$x(a) \quad \text{et} \quad \dot{x}(a)$$

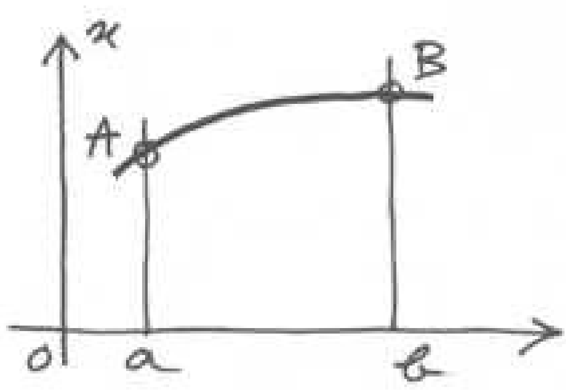
which, graphically speaking, correspond to:



- What bothers you is that you have to determine your solution by imposing the passage through two points, which comes down to the same thing. In a one-dimensional case, we would oblige the curve to pass through two points A and B, which means you have to set:

x(a) and x(b)

and graphically speaking:



- Yes, you are right, it is equivalent. The solution of a differential scalar equation of the second order depends on two parameters. You can determine these two parameters with the help of the ***initial conditions*** x(a) and x'(a) or by using the ***limit conditions*** x(a) and x(b).

- So that will be the same as n dimensions.

- Yes. I just have to get used to the idea that we determine the solution by making the curve x(s) pass through two points A and B, that is to say we take 2n values:

$$\left(x_1(a), x_2(a), \cdots, x_n(a)\right) \quad et \quad \left(x_1(b), x_2(b), \cdots, x_n(b)\right)$$

- I would now like to return to this space that you called ***Phase Space.*** If I had understood correctly, this space has 2n dimensions:

$$(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$

- If that bothers you, you could simple set out:

$$\begin{cases} y_1 = \dot{x}_1 \\ y_2 = \dot{x}_2 \\ \cdots \\ y_n = \dot{x}_n \end{cases} \rightarrow \underbrace{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)}_{2n \text{ dimensions}}$$

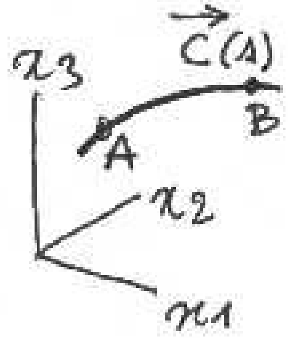
That would allow you to put your Lagrangian in this space:

$$L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$

$$L(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

Would could equally write the actions:

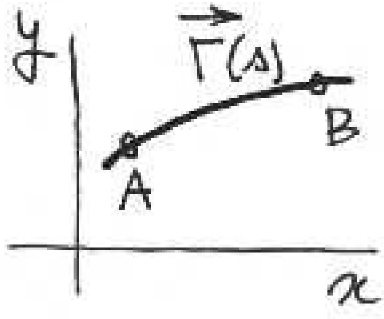
$$A(\vec{C}) = \int_A^B \mathcal{L}(\vec{x}, \dot{\vec{x}})\, ds$$



By associating a mental representation in the form of a "trajectory" **C**(s) in the space **x**.

Or :

$$A(\vec{\Gamma}) = \int_A^B \mathcal{L}(\vec{x}, \vec{y})\, ds$$

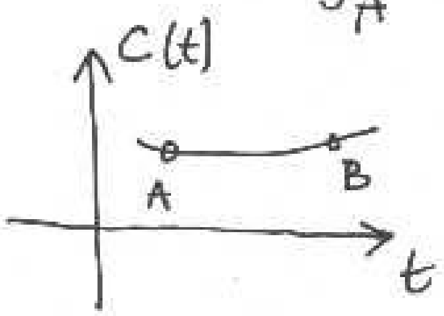


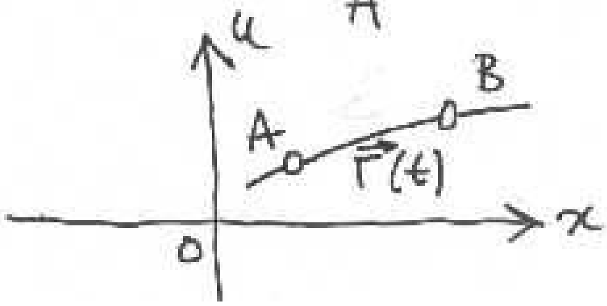
and associate with it a mental image of the "trajectory" **Γ**(s) in the phase space.

- I get the impression that I'm getting it now, providing that I consider s to be time.
- Then you need to make a dynamic, a kinematics, of a point.
- So I can write :

$$s = t \ \textbf{(time)} \qquad \dot{x} = \frac{dx}{dt} = u$$

$$A(C) = \int_A^B \mathcal{L}(x, \dot{x})\, dt = \int_A^B \mathcal{L}(x, u)\, dt$$





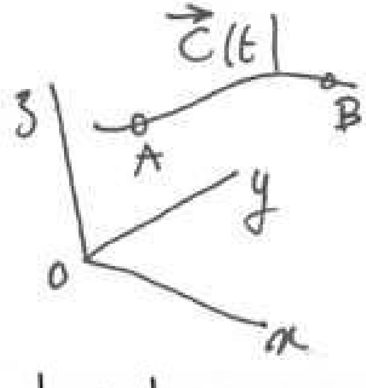
On the left I have a trajectory **C**(t) in a space-time (x,t). On the right I have my trajectory **Γ**(t) in a two dimensional phase space (x, u), configured by my time t. But why limit myself to one dimension of space? I could have three. If I call (x , y , z) my space coordinates and if I set s = t (time) I would have :

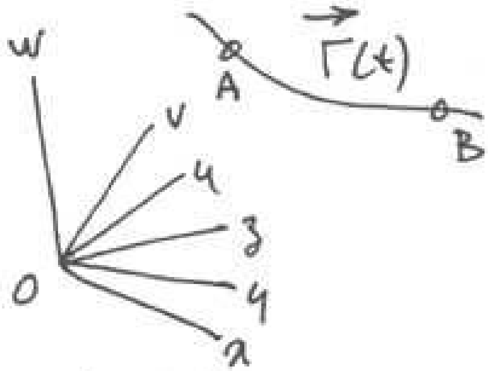
$$s = t = \text{time} \qquad \begin{cases} \dot{x} = \dfrac{dx}{dt} = u \\[2mm] \dot{y} = \dfrac{dy}{dt} = v \\[2mm] \dot{z} = \dfrac{dz}{dt} = w \end{cases}$$

where (u , v , w) represents my speed **V**. And; like earlier, I can symbolize the action in two different ways (but, in any case, there is only one integration variable s or t):

$$A(\vec{C}(x,y,z)) = \int_A^B \mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z})\,dt$$

$$A(\vec{\Gamma}(x,y,z,u,v,w)) = \int_A^B \mathcal{L}(x,y,z,u,v,w)\,dt$$





trajectory in the (x,y,z) space

trajectory in the phase space

I could give myself a mental image of my trajectory as either C or G, the first being inscribed in a space

$$\left(x, y, z\right)$$

of three dimensions and the second in my phase space

$$(x , y , z , u , v , w)$$

of six dimensions. The Lagrange equations form a differential system of the second order in the space (x,y,z) because they bring in secondary derivations and a system of the first order if we consider them to be a differential system on the phase space (x , y , z , u , v , w).

But this brings up a problem.

- What?

- What is the mental universe of a bat?

- What do you mean?

- You know that the bat is practically blind: it "sees" with its ears. It emits ultrasounds with the help of its nose, and then captures them with its large auricular pavilions, constructed like radar antennae or radio telescopes. Its eardrums, more sophisticated than ours, act as retina. It has "biauricular" vision.

- In sum, it "sees" like someone moving in darkness using a torch. But instead of emitting light and picking up the return signal, it emits ultrasound.

- Yes, but you can see the difference: the bat is capable of measuring the Doppler effect in real time. As it has two ears, it means it has not only a 3D perception of the position of objects that it "lights up" but also to know their speed vector. So its mental space is a six dimensional space, a phase space.

- Ah yes...

- I have often thought of what I would feel if I were a bat. One could have fun imagining its perceptive universe with its fake colours. If we suppose that it emits in certain frequency band, we can liken it to colours: blue towards the high frequencies, red towards the low. A smooth object, highly reflective, would seem to be a "white surface", a window for example. Inversely, a fur coat on a sofa would be "black". The reflective power of different objects gives them different "colours". These will then be subtlety modified because of the Doppler effect. Everything getting further away will be a "bit redder" and that which is approaching will be "moved towards blue". In fact, what we cannot imagine is the conception of the in real-time of the group position-speed. The bat has a hexadimensional representation of its world space.

- It lives in an Euclidean space of six dimensions.

- This is very important in its search for its favourite prey: nocturnal moths. Do you know how they try to escape?

- They are covered with absorbent material, hairs and scales on their wings.

- But when they feel that they have been "spotted", like a plane caught in a radar beam, they can sometimes drop as if they were a falling leaf. The bat is sometimes caught out. But there is something even more fascinating. Some nocturnal moths are equipped with organs which allow them to use countermeasures: they emit random, anarchic ultrasounds, which completely upset the bat's mental image, both for position and speed.

- They jam the signal then.

- Exactly. As well as that the bat has to open its mouth wide to catch them. When it does so its ears fold backwards.

- In short when it strikes, it becomes blind.

- If it makes a mistake on the exact position and on the direction of the moth it hits it but misses. But in fact it is even more complex; the moth can artificially modify its ultrasonic signature.

- Camouflage?

- By sending ultrasounds ad hoc it can try and pretend that it is an inedible species.

- If the speed of light was, let us say ten or twenty metres per second, we would also perceive the world as a six dimensional space.

- Terrifying. But if we say that, intelligent species with nocturnal habits wouldn't have developed senses of this type.

- You imagine humanoids functioning like bats, with enormous ears and a fairly complicated nose?

- Why not?

- In that case they would be born with the phase space inscribed in their neurones. Turyshev, you have a great imagination.

- No, I am simply trying to imagine an eventuality, that's all.

- OK, let's return to our business. We said that the Lagrangian "lived" in phases, that it was a differential function for this species.

$$L (x_1 , x_2 , ..., x_n , y_1 , y_2, ..., y_n)$$

I now propose that we consider another function defined in this space and constructed from our function L. let us say:

$$E(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = E(x, y)$$

$$E(x_1, x_2, x_3, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$

$$E = \sum_i y_i \frac{\partial L}{\partial y_i} - L = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

We will see that this new function possesses remarkable properties.

Remember that our function L, the Lagrangian, allowed us to build a particular trajectory AB in the space of all the curves joining two given points in space (x_1 , x_2 , ..., x_n). This particular curve is what makes an action extremal and which is simply the integral of the Lagrangian :

$$\text{Action} : A(\vec{x}) = \int_a^b \mathcal{L}(\vec{x}, \dot{\vec{x}})\, ds$$

I propose to show you now that this function E is constant along the extremal path. Mathematicians call it a ***first integral***. Go on, it is very easy.

- OK, I write :

$$\frac{dE}{ds} = \sum_i \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} + \sum_i x_i \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_i} - \frac{dL}{ds}$$

$$\frac{dL}{ds} = \sum_i \left[\frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i \right]$$

$$\frac{dE}{ds} = \cancel{\sum_i \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i}} + \sum_i \dot{x}_i \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_i}$$

$$- \sum_i \frac{\partial L}{\partial x_i} \dot{x}_i - \cancel{\sum_i \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i}}$$

$$\frac{dE}{ds} = \sum_i \dot{x}_i \underbrace{\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} \right)]}_{\text{Lagrange} \to 0}$$

$$\frac{dE}{ds} = 0 \qquad E \text{ is constant}$$

- You are right, it isn't very complicated.

- Do you know what this function represents?

- No.

- Energy.

- Fancy that!

- To conclude, understand that this equation, set out at the end of the... eighteenth century by the mathematician Lagrange, is the breviary of physicians, theoretical physicians and geometers. You'll be amazed how many more rabbits we can pull out of this hat.

Boissinière brought the meeting to an end.

- Coincidence, I think that we have rabbit for dinner, though frozen of course. I think it's time to rejoin the others in the canteen.

After dinner, Bourbakof decided to "get some fresh air on the bridge", which could only be fictitious as it was inside the ship. He climbed the stairs. At the top there was a vaulted ceiling covered in stars. Someone had already been there and put it in "exterior" mode. He let his eyes get used to the obscurity before noticing Turyshev sitting on a deck chair.

- Amazing don't you think? We could imagine ourselves on the top deck of a French Line ship. If Boissinière had installed a ventilator with a bit of imagination we could imagine ourselves on the high seas.

- True enough. It's a pretty unusual "upper deck" though with Jupiter behind us.

- Yes I set the viewpoint so that we wouldn't be dazzled by the planet. But if you want to see it...

- No, I saw it yesterday in full view. I prefer the peace of this starry ceiling.

- The stomach of the goddess Nout.

- We will put ourselves in her hands. Do you know where we are going?

- No, nor does Fowler. I suppose that Boissinière must know where he's going, but in the end, what does it matter? We are going somewhere, that's all...

- I hope I didn't make you suffer too much this afternoon with the lesson.

- On the contrary, in the future I will be a keen student of all the seminars you have the kindness to give us. I found it very amusing to calculate the gradients in six dimensional space. Though I'm a biologist I might well have enjoyed being a physician. What fantastic tools! A Lagrangian, a bit of variational calculation and suddenly we have control of the soap bubble problem.

- The world isn't made of soap bubbles.

- Firstly, I think that with a Lagrangian there must be lots of other things that we can do besides calculations, and secondly, I don't agree with you. Soap bubbles are very important. Everything that concerns a dream is important.

- Why did you join the group Turyshev?

- In the beginning I was working on AIDS. We were trying to see if we couldn't get results using microwaves.

- You mean by heating tissue?

- No. The AIDS virus has a particularity that makes it very dangerous: it shelters in police stations, so to speak, as it lives inside lymphocytes. We thought that by using HF, in the gigahertz band, the cells would be become relatively transparent when subjected to these frequencies.

- So what interest is there in that?

- Did you know that DNA and RNA are 3400 times more absorbent than water when they are subjected to HF modules at very low frequencies?

- No, I didn't know, but I get the idea. The cells' cytoplasm are transparent with high frequencies but the long molecules of this retrovirus then play the role of antennae and are sensitive to low frequencies. I imagined that you envisaged damaging the RNA of the virus while using relatively low energies...

- That is the idea. But the research attracted all sorts of undesirable people. You remember the phrase in the Bible, in Geneses, where God forbids Adam and Eve to touch the tree of life?

- There was the tree of knowledge of good and evil whose fruit they ate, then this tree of life, protected by angels, who forbid access to it. I've often wondered what it was.

- Genetics my friend. We are in the act of touching the tree of life as complete ignorants. With pulsated microwaves we can not only break up the virus, we can also modify it.

- Why is genetics so little known?

- Maybe we haven't wanted to touch such things too early. It is maybe simply a question of time.

- We are totally ignorant in this area. Did you know that if a certain sequence is present in the genome of a child, the child will contract glaucoma and become blind but that if this sequence is present twice he won't contract it. Do you have any explanation for such a strange thing?

- Most certainly not.

- Quite, and when one doesn't understand something one takes care not to touch it, especially if it is wearing a uniform and stripes and we want to obtain new biological armaments and artificially mutated virii.

- That is when you left the Collective?

- You've understood completely. They are everywhere. In every field the Collective acts as their valet.

- Do you think that this spaceship will help us avoid such errors?

- I can only hope so, if not I will ask to be let off on the first planet we come to.

- If that is the case there will be two of us.

- Then you can give me maths lessons...

Lagrange & Newton

Fowler rejoined the group in the seminar room.

- You know Bourbakof; your seminars are highly appreciated. Boissinière set up a webcam last time. Everyone followed it closely even though many of them were at their control positions.

- You are too kind.

- These little mathematical amusements very formal, are very welcome to keep up morale during our voyage to goodness know what destination on board this camembert propelled by goodness knows what.

Boissinière interrupted:

- Don't be upset by the presence of the propulsor Without it we would be at acceleration zero and would be floating in the corridors like simple cosmonauts.

- Heavens no, Fowler protested! At the time of our departure, when we cut off the MHD after leaving the atmosphere when we were in weightlessness for several dozen seconds, I almost brought up the sandwich that I swallowed just before boarding the ship.

Very ceremonially Boissinière turned towards Bourbakof:

- My friend, you are the boss. It is up to you. Couldn't you pull a Lagrangian out of your hat in order to apply all those marvels you evoked earlier?

- I was just thinking of that.

Bourbakof went up to the blackboard slowly, took the chalk and wrote:

$$L(r, \dot{r}, \cancel{\theta}, \dot{\theta}) = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{A}{r}$$

before adding: "A is a constant".

Boissinière took Turishev's place:

- There is no reason why it should always be the same people who have the fun. Now it is my turn to play the maths student with a sticky problem.

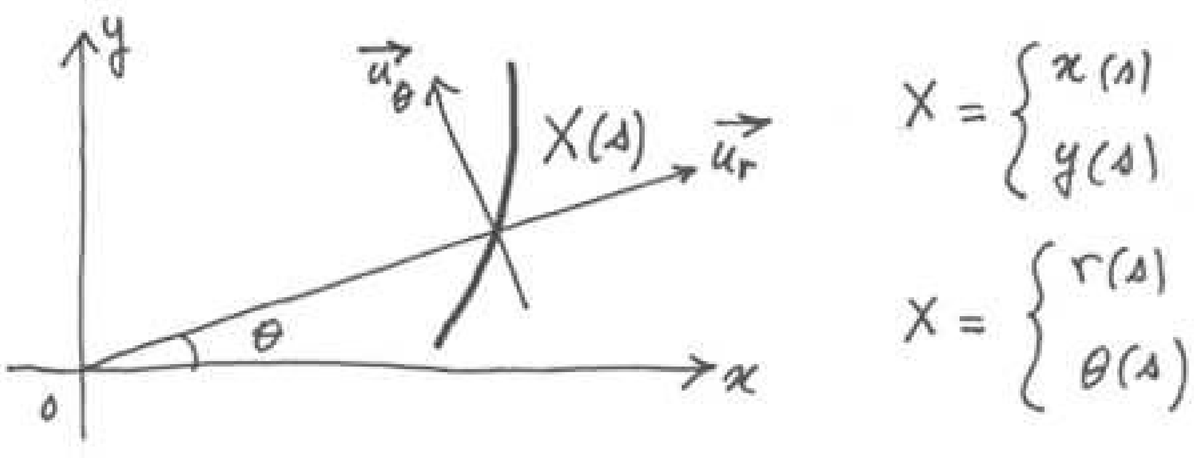
- What have you noticed, Bourbakof asked?

- That your Lagrangian, which corresponds to a two dimensional space rand θ, does not depend on θ.

- And that means?

- Let us see, your little game resembles something that plays on polar coordinates.

- You could put it that way...



- The problem consists of the construction of two function solutions r (s) and θ (s). But if I suppose that we try to put this solution in the form of a function (θ) or the inverse θ (r). If θ is not preset in your Lagrangian then if I have a function solution r (θ), then every other function r (θ + α) will also be solution.

- In short, we will have symmetry.

- A symmetry in relation to what? ...

- Ah, always the same problem with physicians! The word symmetry doesn't cover the same notions for you and for us. A physician does not know symmetries in relation to a straight line, a plane or a point. Mathematicians can adapt lots of other things to the word: the simple fact of being invariable in relation to a certain action.

- You mean ... the action of turning?

- Precisely. It is question of symmetry of rotation.

- I'm not sure I understand you...

Fowler intervened:

- But of course, my dear Hubert, the fact that something is conserved by rotation, for a mathematician this is symmetry. It would be the same for a translation.

- Really! ....

- Mathematicians have the art of turning the meaning of words back to their primitive sense. It's a real sport with them. If Bourbakof continues to initiate us in his art, you'll no longer notice. Why do they do it? We can only guess. It's possible that they wanted to put a barrier of unintelligibility between them and rest of humanity in order to protect their knowledge, as the alchemists used to do during the Middle Ages. Whatever, it works. It seems that they do not even understand each other. My wife Sarah, who was a psychiatrist, said that they were all schizophrenics. Oh, sorry Bourbakof, obviously I wasn't talking about you.

Fortunately, no-one could resist Fowler's laugh. He was someone without any trace of nastiness. He continued in the same vein:

- Bourbakof? He has a good heart; you can see that straight away. The proof? We all understood what he was saying.

- Even the biologist that I am understood! Turyshev added.

Fowler wagged his finger:

- I wouldn't say as much for the theoretical physicians. They are at the limit of autism. They are really nasty and badly intentioned.

He turned towards Boissinière:

- Perhaps you knew Souriau as you are French?

- Didn't he publish a work called "The Structure of Dynamic Systems»? In 74 I believe. I could never get past the first page.

- No, you need to have spoken "Sourian". He was a bit special. Not only was he a mathematician but he also elaborated a mathematical language for his personal use, as if things weren't complicated enough already.

Bourbakof felt that he needed to intervene.

- He was one of the rare examples of a mathematician who also looked at physics problems and who brought many new things to the field. In fact he fundamentally rethought theoretical Mechanics.

- Theoretical physics? Turyshev asked

- Nothing to do with it, exclaimed Fowler. I always liked Souriau's definition.

- His definition of theoretical physics?

- Yes. He said that it was the intersection of two ensembles: mathematics without the rigour and physics without the experience.

- In that case, said Turyshev intrigued, what is mathematical physics?

- Bourbakof will explain. We have all the light years we want in front of us to answer this question.

Bourbakof smiled.

- I propose that we return to the Lagrangian.

Boissinière, taking up his idea of polar coordinates once more had written:

$$V_r = \dot{r} \qquad\qquad L = \frac{1}{2} V^2 + \frac{A}{r}$$

$$V_\theta = r\dot{\theta}$$

- The first term of your Lagrangian evokes the kinetic energy particle of mass unity.

- Carry on; write your Lagrange equations...

Boissinière did so.

$$\frac{\partial L}{\partial \dot{r}} = \dot{r} \qquad \frac{\partial L}{\partial r} = r\dot{\theta}^2 - \frac{A}{r^2}$$

$$(1) \qquad \ddot{r} = r\dot{\theta}^2 - \frac{A}{r^2}$$

$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \qquad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$(2) \quad r^2\dot{\theta} = h$$

Bourbakof interrupted.

- We will resolve this system more elegantly by using the property that we established at the end of the last session, in which there exists a function E which remains constant along the path solution. Over to you Boissinière.

- I will use the definition of this function E, defined on the phrase space

$$E = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

which gives :

$$E = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \quad \text{is constant}$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 - \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{r}$$

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{r} = \text{constant}$$

- Here, a second trick, we are going to go from the magnitude r to it inverse u:

$$u(\theta) = \frac{1}{r(\theta)}$$

Go on then, Boissinière

- OK, now I replace and integrate.

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

but $r^2 \dot{\theta} = h \longrightarrow \frac{du}{d\theta} = -\frac{\dot{r}}{h}$

$$\frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) = \frac{A}{r} + E$$

$$\frac{1}{2}\left(\dot{r}^2 + h^2 u^2\right) = Au + E$$

$$\dot{r} = -h\frac{du}{d\theta} = -hu'$$

$$\dot{r}^2 = h^2 u'^2$$

$$\frac{1}{2}\left(h^2 u'^2 + h^2 u^2\right) = Au + E$$

$$u'^2 + u^2 = \frac{2Au}{h^2} + \frac{2E}{h^2}$$

$$u' = \pm\sqrt{-u^2 + \frac{2Au}{h^2} + \frac{2E}{h^2}}$$

which gives us finally :

$$d\theta = \frac{\pm\, du}{\sqrt{-u^2 + \frac{2Au}{h^2} + \frac{2E}{h^2}}}$$

$$\theta = \pm \int \frac{du}{\sqrt{-u^2 + \frac{2Au}{h^2} + \frac{2E}{h^2}}}$$

Boissinière stood back, crossed his arms and contemplated his calculation.

- OK, what do I do now?
- You change the variable to show:

$$\sqrt{-X^2 + 1}$$

which, when integrated, gives you an Arc cosine

- Bourbakof, it has been at least twenty years since I last did an integral calculation! .....

Bourbakof sighed.

- So be it, but take me at my word and do it.
- OK ..., we write :

$$-u^2 + \frac{2Au}{h^2} + \frac{2E}{h^2} = -\left(u - \frac{A}{h^2}\right)^2 + \frac{A^2}{h^4} + \frac{2E}{h^2}$$

which brings us to :

$$x = u - \frac{A}{h^2} \quad et \quad c^2 = \frac{A^2}{h^4} + \frac{E}{h^2}$$

and which brings our integral to:

$$\theta = \pm \int \frac{dx}{\sqrt{-x^2 + c}}$$

a second variable change :

$$\nu = \frac{x}{c} \qquad dx = c\, d\nu$$

will give us finally :

$$\theta = \pm \int \frac{d\nu}{\sqrt{-\nu^2 + 1}}$$

$$\theta = \mathrm{Arcos}(\nu) = \mathrm{Arcos}\left(\frac{u}{c}\right)$$

and to conclude :

$$\cos\theta = \frac{1}{c}\left(\frac{1}{r} - \frac{A}{h^2}\right)$$

- And what is this thing?

- It's the representation of a conic in a polar.

- Ah yes, I remember this vaguely. But once more, I'll take you at your word.

It was time for coffee which brought everyone around the ineffable machine on the ship. All except two: Turyshev, who wanted to know everything, wanted an explanation as to why the equation described a conic. When everyone returned he was the one to ask the key question:

- OK. Bourbakof threw us a Lagrangian, like drawing a rabbit out of a hat:

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{A}{r}$$

The variable is time t. So it is question of the mechanics, the dynamics of a material point. The calculation gives the trajectories in the form of conics. Question: what is this 'underlying dynamics"?

- I'm getting there, said Bourbakof, I'm getting there. Let us return to the first of the two Lagrange equations:

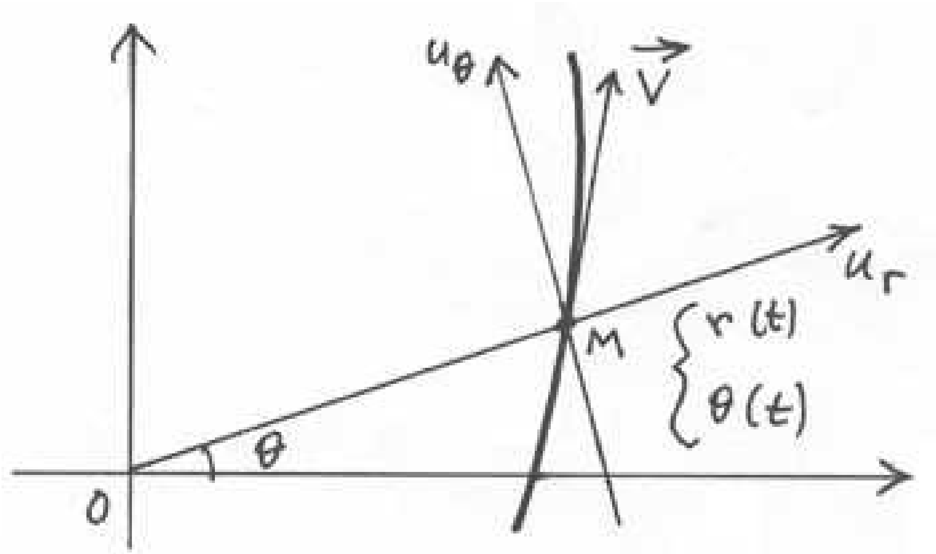
$$(1) \qquad \ddot{r} = r\dot{\theta}^2 - \frac{A}{r^2}$$

that we shall now write as:

$$(1) \qquad \ddot{r} - r\dot{\theta}^2 = -\frac{A}{r^2}$$

- This shows us something. But we have to find out what. The problem is to find out the signification of the first member.

Let us trace the trajectory in polar coordinates:



$\mathbf{u}_r$ and $\mathbf{u}_\theta$ are unitary vectors (radial and "perpendicular"). $\mathbf{V}$ is the speed vector.

$$\vec{V} \begin{cases} V_r = \dot{r} \\ V_\theta = r\dot{\theta} \end{cases} \qquad \vec{u_r} \begin{cases} \cos\theta \\ \sin\theta \end{cases} \qquad \vec{u_\theta} \begin{cases} -\sin\theta \\ \cos\theta \end{cases}$$

$$\vec{V} = \dot{r}\,\vec{u_r} + r\dot{\theta}\,\vec{u_\theta}$$

Let us calculate the acceleration vector:

$$\vec{\Gamma} = \frac{d\vec{V}}{dt} = \ddot{r}\,\vec{u_r} + \dot{r}\,\frac{d\vec{u_r}}{dt} + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\vec{u_\theta} + r\dot{\theta}\,\frac{d\vec{u_\theta}}{dt}$$

$$\frac{d\vec{u_r}}{d\theta} = \vec{u_\theta} \qquad \frac{d\vec{u_\theta}}{d\theta} = -\vec{u_r}$$

$$\vec{\Gamma} = \ddot{r}\,\vec{u_r} + \dot{r}\dot{\theta}\,\vec{u_\theta} + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\vec{u_\theta} - r\dot{\theta}^2\,\vec{u_r}$$

$$\vec{\Gamma} = (\ddot{r} - r\dot{\theta}^2)\,\vec{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{u_\theta}$$

We have a radial component and a "perpendicular" component. But here we bring in the second Lagrange equation:

$$(2)\ r^2\dot{\theta} = h \rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

So it is question of a central acceleration movement. We know that central acceleration movements are situated in planes that in this case are conics and which correspond to ***Keplerian trajectories.*** The quantity :

$$r^2\dot{\theta}$$

is nothing other than the orthogonal kinetic movement component on the trajectory plane and this quantity is conserved in a central acceleration movement.

This allows us to express the constant A:

$$\vec{\Gamma} = \frac{d\vec{V}}{dt} = -\frac{A}{r^2}\,\vec{u_r} = -\frac{GM}{r^2}\,\vec{u_r}$$

and to rewrite the Lagrangian in the form:

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GM}{r}$$

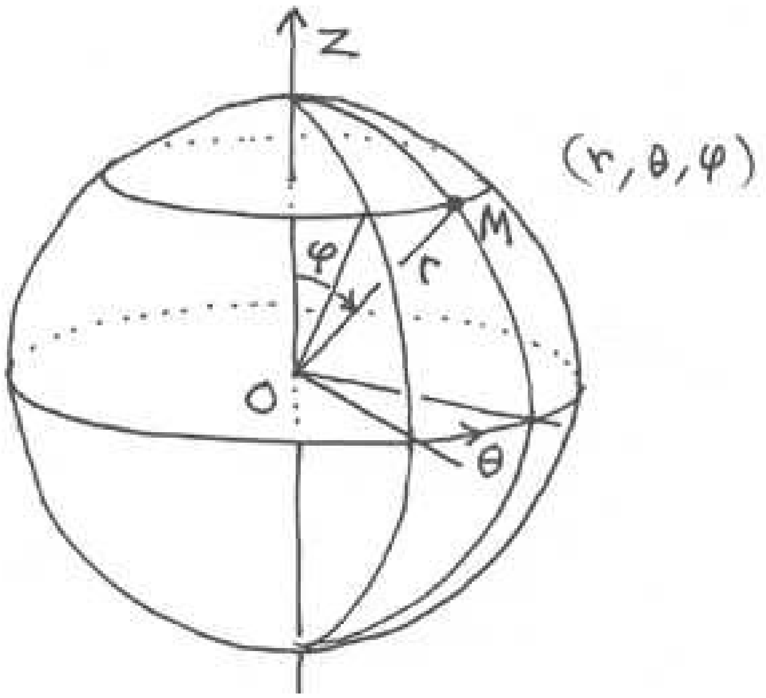
In passing let us return to the energy E:

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{r} = \frac{1}{2}V^2 - \frac{GM}{r}$$

and because of the definition of potential energy, we can see that these problems of scalar mechanic E represent:

E = Kinetic energy + Potential energy

We see that we have completely formulated a problem of mechanics in terms of a Lagrangian. That being said, we are still in 2D. Kepler is 3D. Let us reformulate that with a third dimension. We will have a third coordinate φ which corresponds to the classic spherical coordinates.



So we write the Lagrangian in 3D and the Lagrange equations which follow from this:

$$L = \frac{1}{2}V^2 + \frac{GM}{r} = \frac{1}{2}\left(\dot{r}^2 + r^2 \sin^2\varphi\, \dot{\theta}^2 + r^2 \dot{\varphi}^2\right) + \frac{GM}{r}$$

$$\frac{d}{ds}\left(\dot{r}\right) = r \sin^2\varphi\, \dot{\theta}^2 + r\dot{\varphi}^2 - \frac{GM}{r^2}$$

$$\frac{d}{ds}\left(r^2 \dot{\varphi}\right) = 0$$

$$\frac{d}{ds}\left(r^2 \sin^2\varphi\, \dot{\theta}\right) = r^2 \sin\varphi \cos\varphi\, \dot{\theta}^2$$

So how do we show that the trajectories are inscribed in planes? We will start from a particular solution:

$$\begin{cases} \dot{\varphi} = 0 \\ \varphi = \frac{\pi}{2} \end{cases} \longrightarrow \begin{cases} r(t) \\ \theta(t) \\ \varphi = \frac{\pi}{2} \end{cases} \left\{ r(\theta) \right.$$

The solution then identifies itself with what we have just constructed. But we could consider the values on the left as the initial conditions for the problem of resolution of the system of two differential equations. We will arrive at the same solution because of the concept of unicity: the solution of a differential equation or system of differential equations is entirely determined by the datum of the initial conditions. The solution that will be built with the help of this choice of initial conditions cannot be different from that on the right. Therefore the trajectory is a flat. We obtain a family of conics which all have the origin as one of their focal points. The spherical symmetry of the problem makes possible the deduction of the ensemble of all the solutions. All these trajectories are situated in the planes containing the point O, where the mass M is localized.

Appendices

1. Solving the equation

Here is how to resolve the differential equation that interests us:

$$yy''+1-y^2=0$$

To begin with, notice that if f symbolizes a solution to the equation, $f\,'$ cannot cancel itself out on any interval. If not, $f\,''$ would also be nil on this interval which would lead to the contradiction:

$$1=0.$$

This being so, f is a solution to the equation. We could therefore locally define the inverse function :

$$y=f(x) \quad \Leftrightarrow \quad x=f^{-1}(y)$$

and write :

$$z(y)=f\,'\circ f^{-1}(y)$$

Let us now derive this function z, where we get :

$$z'=\frac{f\,''\circ f^{-1}}{f\,'\circ f^{-1}}$$

or, given the fact that :

$$z'=f\,'\circ f^{-1}$$

We have :

$$zz'=f\,''\circ f^{-1}$$

By using the equation, we get :

$$f\,''\circ f^{-1}=\frac{\left(f\,'\circ f^{-1}\right)^2+1}{f\circ f^{-1}}=\frac{z^2+1}{y}$$

Where :

$$yzz' = z^2 + 1$$

This can also be written as:

$$\frac{zdz}{z^2+1} = \frac{dy}{y}$$

and is easily integrated to give:

$$z^2 + 1 = ay^2$$

So :

$$f' = \sqrt{af^2 - 1}$$

This equation is integrated once more. It can be written more simply as :

$$a = s^2$$

We obtain :

$$sf(x) = ch(s(x+c))$$

There are two integration constants, s and c but if we limit ourselves to even solutions, that is to say by verifying:

$$f(-x) = f(x)$$

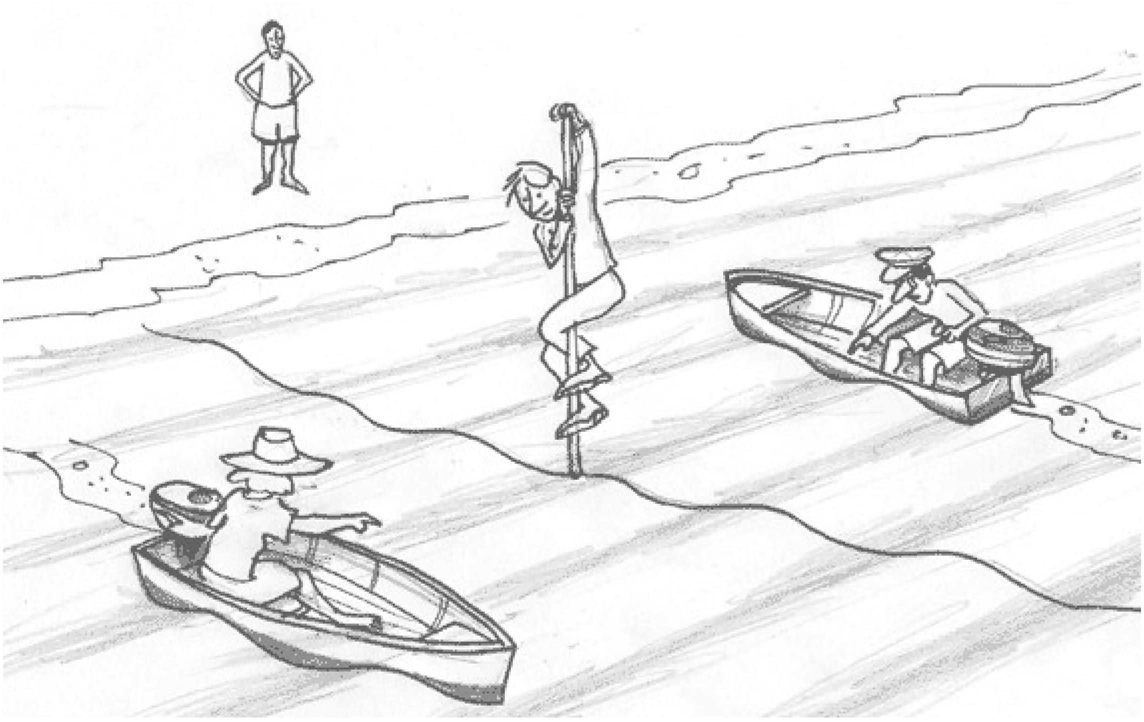
The constant c must be nil and we obtain therefore a family of one parameter solutions:

$$f(x) = \frac{ch(sx)}{s}$$

2. *Doppler Effect*

The noise of an approaching car is at a higher pitch than the noise of a car moving away from the listener. That is an example of the Doppler Effect that everyone can recognize. This “Doppler effect” is found all over the place, in medical imagery for example or in astrophysics where we talk about the "red shift". In fact the Doppler Effect is a universal phenomenon: it is the modification of the frequency of a periodic signal between the emitter of a signal and the receiver of the signal if there exists a relative movement between the two.

Let us be more precise and use an example. Imagine a beach on which waves arrive from out to sea. A little further out there is a pole and on the pole there is an unfortunate observer who has to count the number of crests of waves that pass beneath him. We will symbolize the number of crests observed per unit of time by ν. This number is called the frequency. On either side of our first observer there are two small boats with outboard motors. One moves away from the beach and the other moves towards it. The two helmsmen on the boats are also counting the number of wave crests. The master of the boat moving away from the beach counts the number of waves that break on his bows whereas the other counts the number of waves that hit the stern of his boat. From his point of view the waves are chasing him and the number of waves that reach his boat per unit of time, ν' , is less than the number of waves passing below the pole. However for the boat moving away from the beach, this effect is inverted: the number of waves that hit his boat per unit of time is superior to the frequency observed by the poor fellow up the pole.



Let us examine that more closely. We will call the wave speed c. the observer on the pole sees a series of N crests cross the pole during an interval of time Δt. If the distance between two crests, which we will call the wavelength, is L, then the wavelength of the train of the wave observed during the period of time Δt is NL. We have :

$$NL = c\Delta t \quad et \quad N = \nu\Delta t$$

For the helmsman of the boat going towards the beach the time $\Delta t'$ that this wave train takes to pas the bow of his boat is longer. The speed at which the waves arrive at the boat is less than the speed at which the waves arrive at the pole, it is: c-u where u symbolizes the speed of the boat, so we have:

$$NL = (c-u)\Delta t' \quad et \quad N = \nu'\Delta t'$$

This gives us :

$$NL = (c-u)\Delta t' = c\Delta t \quad et \quad N = \nu\Delta t = \nu'\Delta t'$$

Which gives us :

$$\nu' = \frac{\Delta t}{\Delta t'}\nu = \left(1 - \frac{u}{c}\right)\nu$$

By designating with $\Delta\nu = (\nu - \nu')$ the frequency differences, we get, finally:

$$\frac{\Delta\nu}{\nu} = -\frac{u}{c}$$

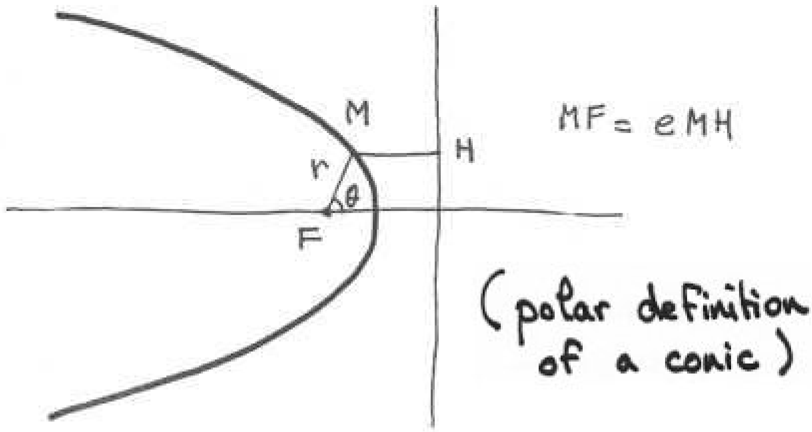
In our example, there are two referentials (three if we also consider the boat moving away from the beach): that of the observer on the pole and that of the helmsman of the boat moving towards the beach. The nominal frequency of the waves N, is that measured by the fellow on the pole. We could consider u, the speed of the boat, to be the relative speed between the emitter (the pole) and the receiver (the boat), that is to say the difference between the speed of the wave train measured by the emitter and that measured by the receiver, It is preferable to symbolize this difference by Δc to obtain finally :

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta c}{c}$$

We call this the frequency shift. If Δc is positive (as is the case for the boat moving towards the beach and which is moving in the same direction as the signal, the wave train) then the frequency shift is negative. The reception frequency is inferior to the emission frequency: the sound of a car moving away is lower, the light of a galaxy 'moving away" is shifted towards the red, which is why we have the expression "red shift" so beloved of astrophysicians. If Δc is negative (as in the case of a boat moving out to sea and going in the opposite direction to the waves) then the observed frequency is inferior: the sound of an approaching car is at a higher pitch

3. Reminder on conics

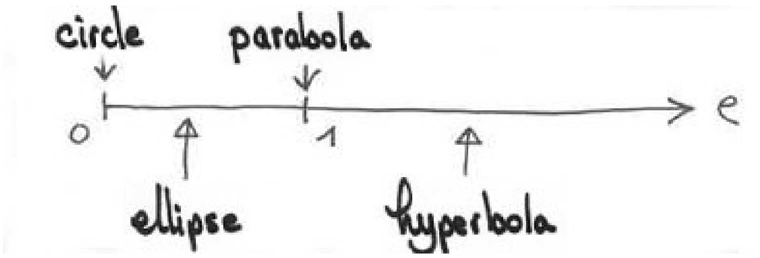
Here is the polar definition of a conic:



$$MH = h - r\cos\theta \qquad eh = p$$

$$r^2 = e^2(h - r\cos\theta)^2 \qquad r = \frac{eh}{1 + e\cos\theta}$$

e is the eccentricity of the conic. The various conics correspond to the following values:



Here are various curves for h = 1 and the e variable.