Appendix 1
GOD'S POLYHEDRON

Science in our times is extremely mediatized. As soon as we evoke an idea, a project, we quickly give it a touting name which will grasp people's imagination. Fifty years ago, the object which we imagined could describe the destiny of a neutron star which mass, because of the influx caused by the stellar wind originating from a companion star, could exceed the critical value of 2.5 solar masses, was called "SCHWARZSCHILD'S BODY" (*). Not a very selling name. The word "COLLAPSAR" didn't have much success either. But when John Archibald Wheeler proposed "BLACK HOLE", its success was immediate and worldwide. Same thing for TOE (Theory of Everything), the "M THEORY" from the superstrings people. Currently our modern plutophysicists (from ploutos which signifies "wealth" in Greek) are searching for the Higg's boson, already nicknamed "GOD'S PARTICLE".

To go along this imbecile fashion and make you smile, here's the polyhedron which has only one face and one edge. We remind you that "hedron" in Greek means "face".

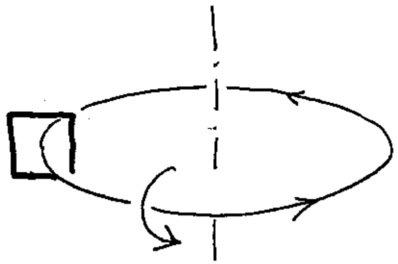
So here's the MONOHEDRON or ... "GOD'S POLYHEDRON".

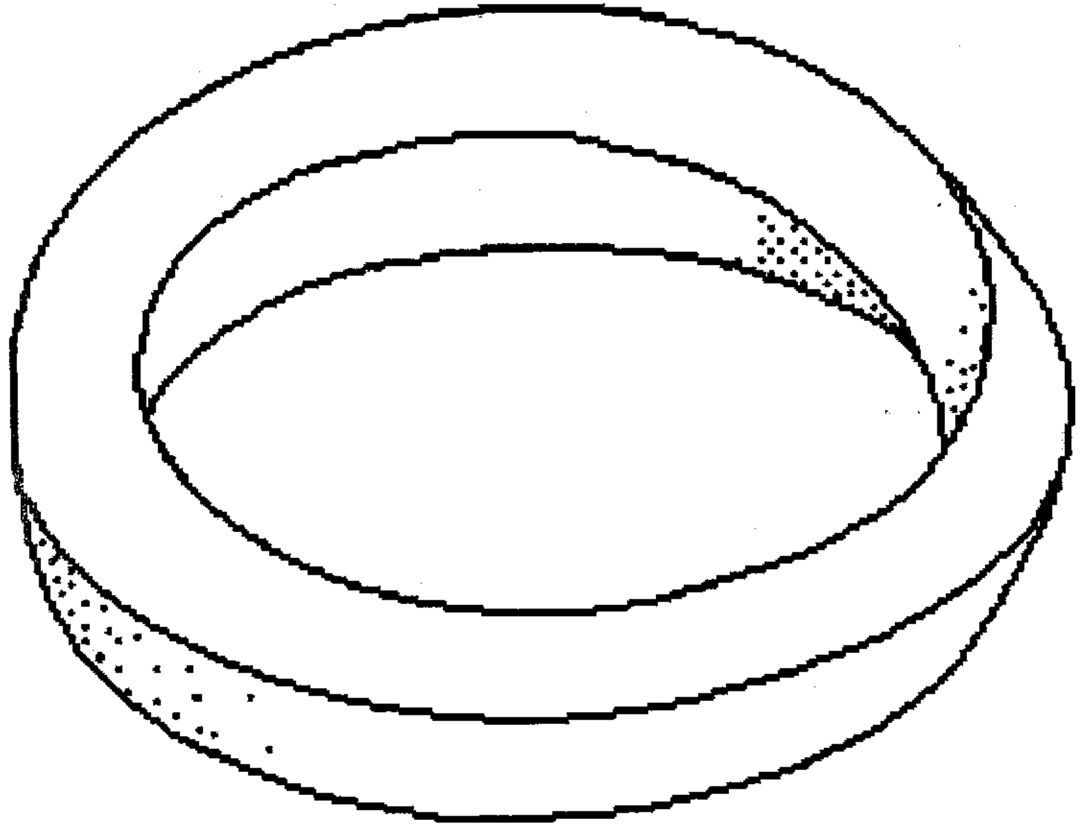
The management

(*) The model of a "black hole" is based on a handiwork from a solution of Einstein's equation, from Schwarzschild (1917), referring to an EMPTY region of the universe. We'll talk about it in a later album.

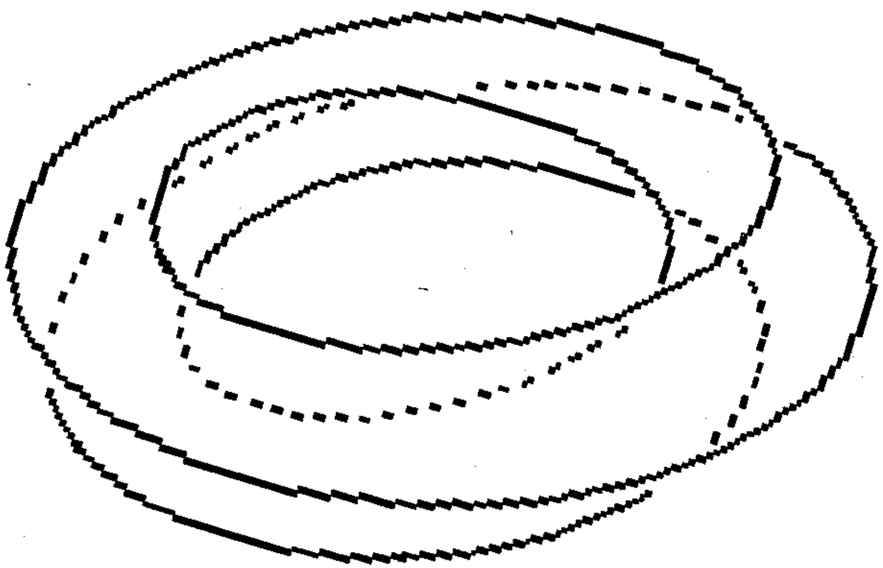
THE MONOHEDRON

We can generate it by revolving a square around an axis contained within its plane and rotating it by $\pi/2$ at each turn.





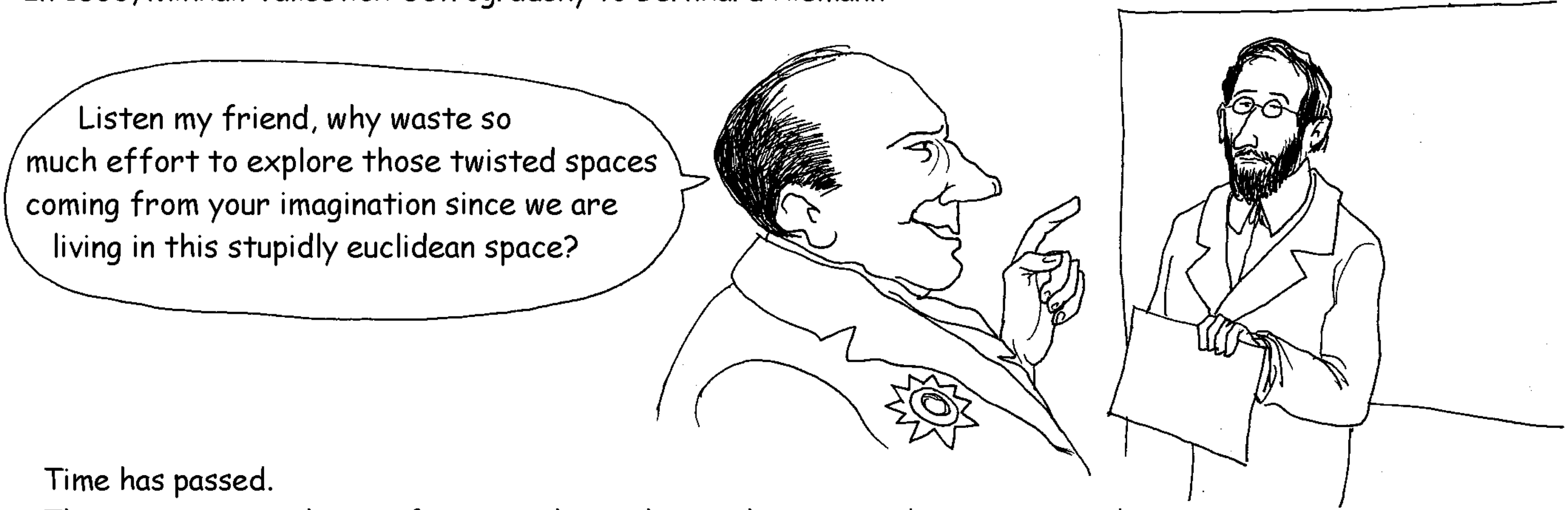
... or by thickening the Möbius ribbon.



ITS UNIQUE EDGE

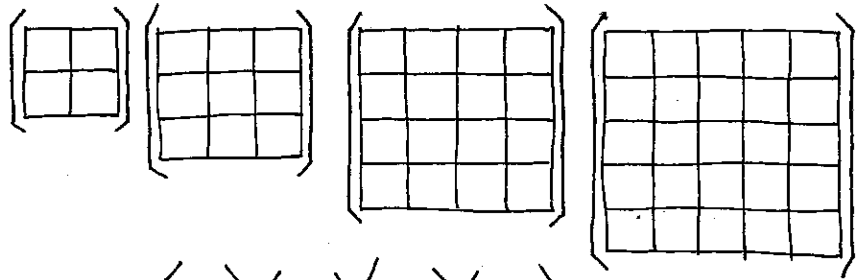
APPENDIX 2
SPACETIMES AND GROUPS

In 1850, Mikhail Valisevich Ostrogradsky to Bernhard Riemann

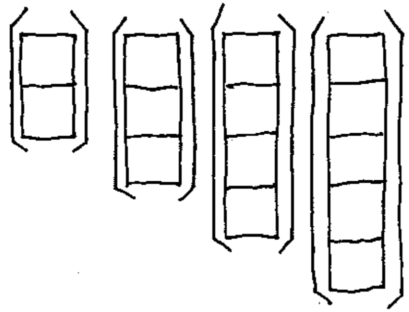


Time has passed.
The permanent evolution of science shows that each time an advance occurred, it was done by abandoning some naive vision coming from our senses. Facts show that mathematicians, especially the geometers, always had a vision of things which revealed itself closer to the experiences of physicists and observations of astronomers than earlier visions which eventually fell into obsolescence. By manipulating new concepts with pencil and paper they create, perhaps without realizing it, the reality of tomorrow. To understand for example SPECIAL RELATIVITY, you must make an effort to do a real LET GO your vision of the world.
Are you ready to follow me ?

The letter M will designate a square MATRIX (n lines, n columns)

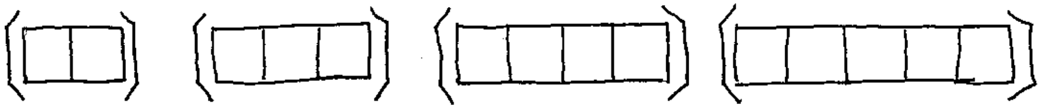


A COLUMN VECTOR is a matrix with n lines and 1 column :



etc...

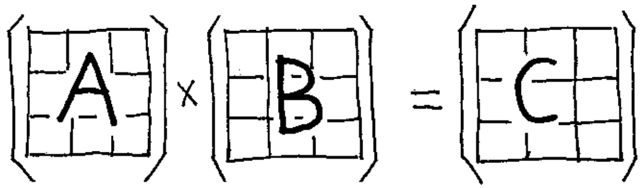
A LINE VECTOR is a MATRIX with 1 line and n columns :



etc...

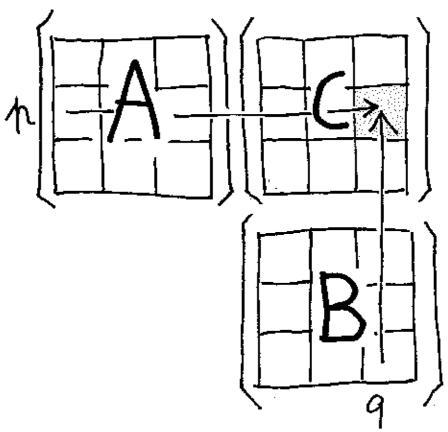
MULTIPLICATION OF TWO SQUARE MATRICES WITH THE SAME FORMAT

(having the same number of lines = number of columns)



$C = A \times B$

we multiply "LINES-COLUMNS"



Mnemonic technique : we place the two matrices A and B of the product matrix A x B as shown on the left and we multiply elements by elements, by adding the elements of the line p of the matrix A by the elements of the column q of the matrix B. This way we obtain the element on the p^{th} line and q^{th} column of the matrix $C = A \times B$.

FUNDAMENTAL: THIS PRODUCT IS NOT, IN GENERAL, COMMUTATIVE.

$$\boxed{A \times B \neq B \times A \quad !}$$

IDENTITY MATRIX I

For every set of square matrices with n lines, n columns [we say "of format (n,n)"] we associate an identity matrix, denoted by I

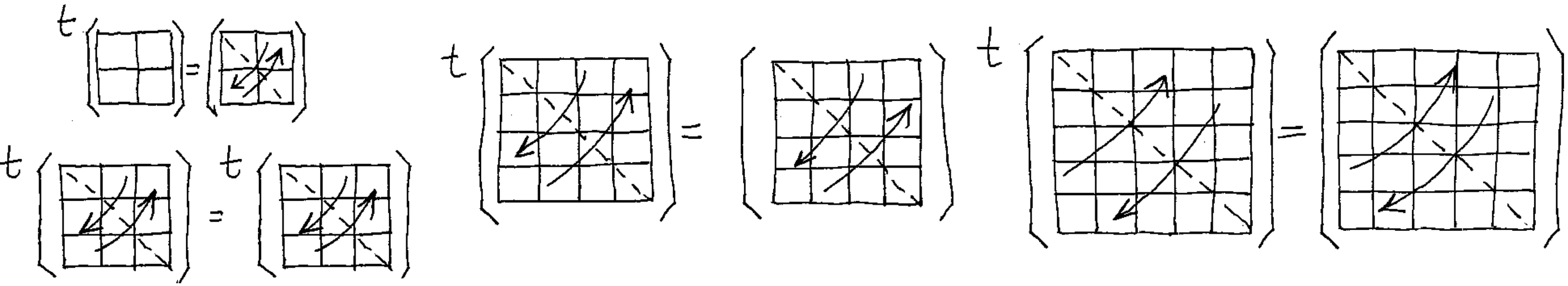
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & {} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{etc...}$$

We have :

$$\boxed{A \times I = I \times A = A}$$

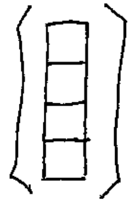
TRANSPOSE OF A MATRIX, DENOTED ${}^{t}A$

It is the symmetric inverse of the square table with respect to its MAIN DIAGONAL.

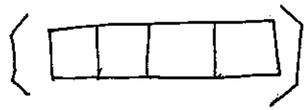


etc...

WE WILL POSE that the transpose of a vector, or column matrix :

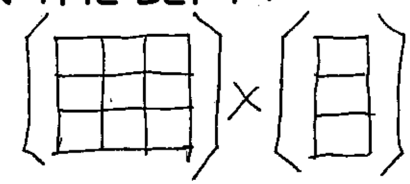
$X =$ 

is the corresponding line matrix :

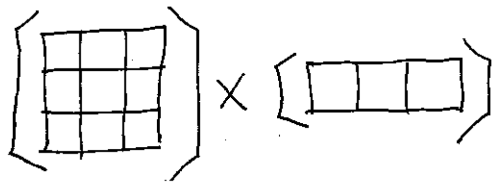
${}^tX =$ 

MULTIPLICATION OF A LINE OR COLUMN MATRIX BY A SQUARE MATRIX

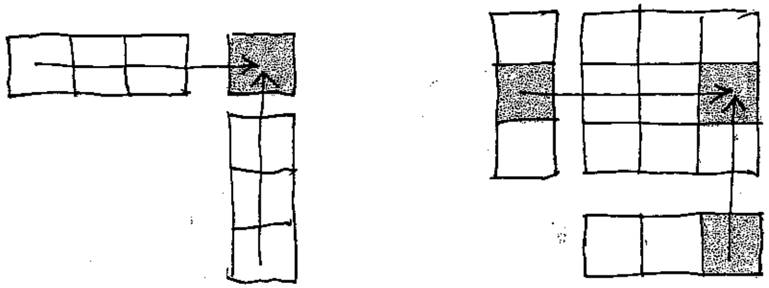
For the column matrix, MULTIPLY ON THE LEFT :

$A \times X =$ 

For the line matrix, MULTIPLY ON THE RIGHT :

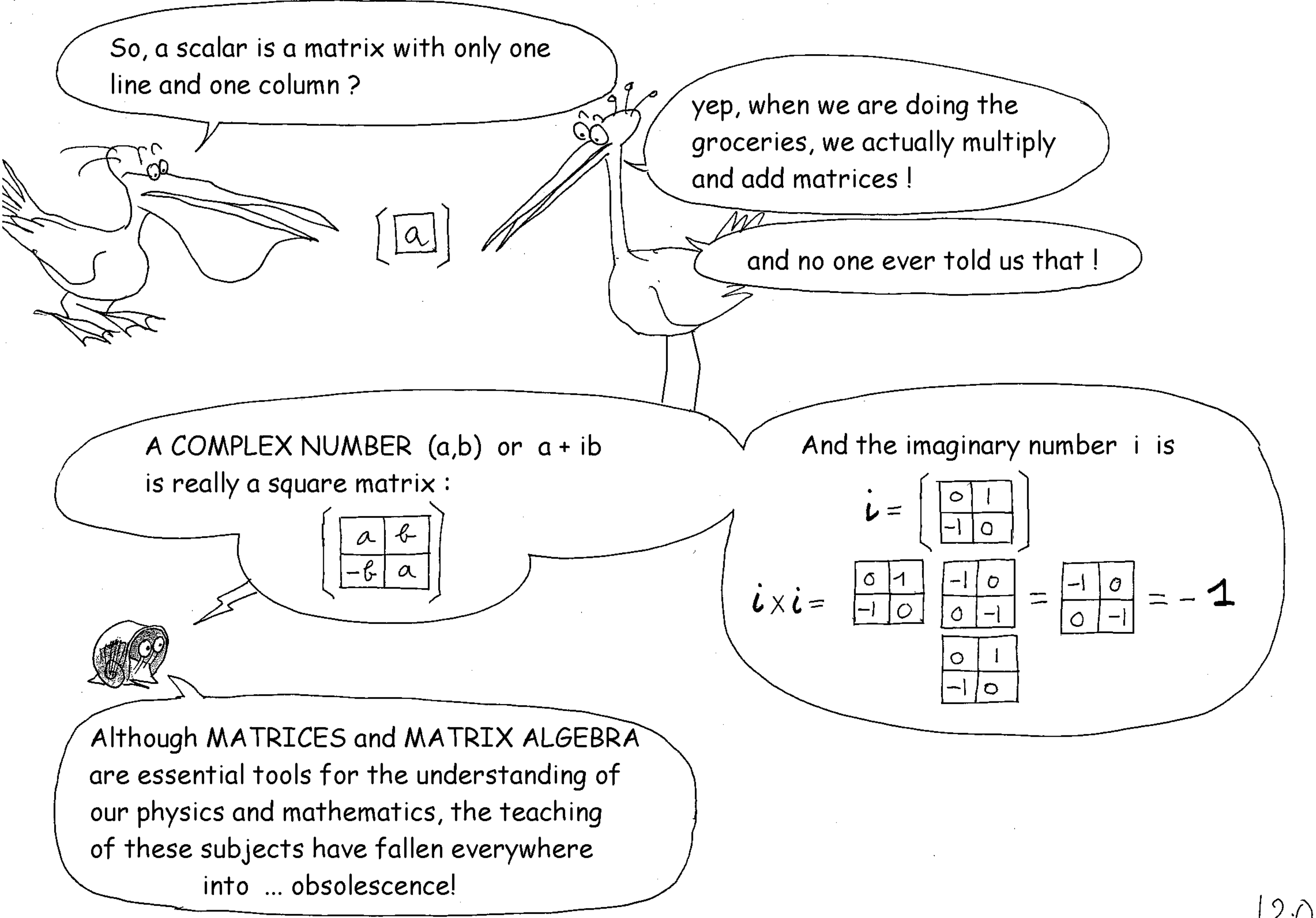
$A \times {}^tX =$ 

PRODUCTS OF A COLUMN MATRIX ⇄ AND OF A LINE MATRIX :



${}^tX \times X =$ matrix with 1 line, 1 column = SCALAR

$X \times {}^tX =$ square matrix with format (n,n)


So, a scalar is a matrix with only one line and one column ?
[a]
yep, when we are doing the groceries, we actually multiply and add matrices !
and no one ever told us that !
A COMPLEX NUMBER (a,b) or a + ib is really a square matrix :
$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$
And the imaginary number i is
$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$i \times i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1$
Although MATRICES and MATRIX ALGEBRA are essential tools for the understanding of our physics and mathematics, the teaching of these subjects have fallen everywhere into ... obsolescence!

Square matrices can have an inverse, denoted A^{-1} such that :

$$\boxed{A^{-1}\times A = A\times A^{-1} = I}$$

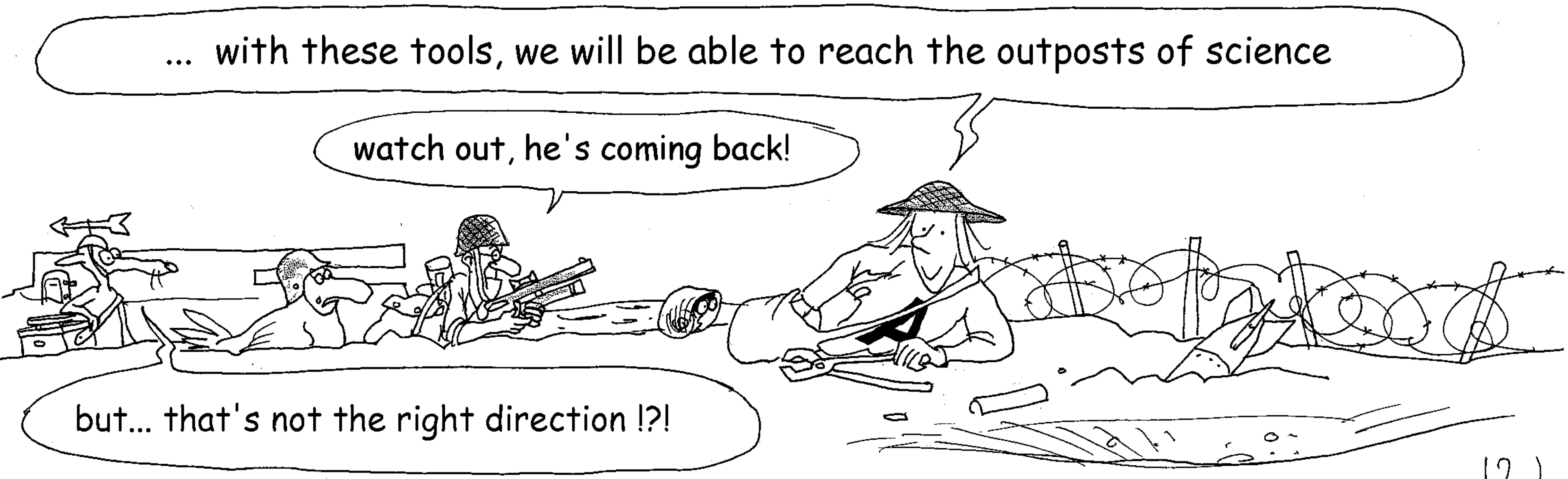
A first theorem, without proof:

$$\boxed{(A\times B)^{-1} = B^{-1}\times A^{-1}}$$

A second theorem, without proof:

$$\boxed{{}^t(A\times B) = {}^tB\times {}^tA}$$

the proofs are easy but without much interest (if you really want to...)



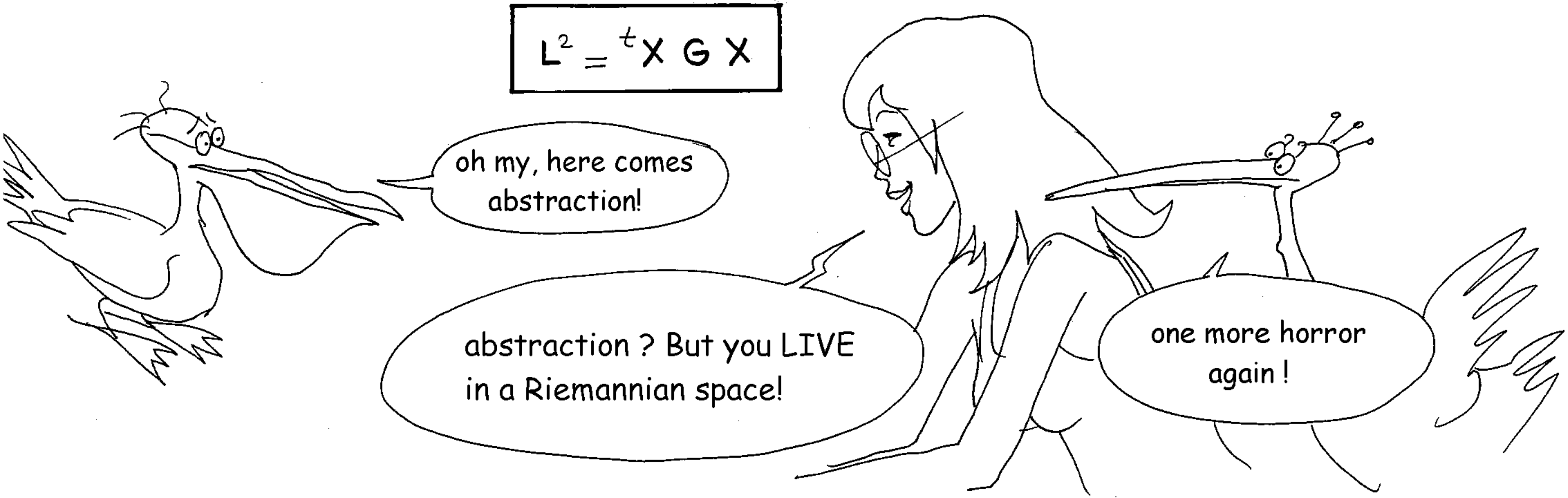
RIEMANNIAN SPACES (*)

We will call GRAM MATRICES square matrices in which all non-diagonal elements are zero and all elements of the MAIN DIAGONAL have value ±1.

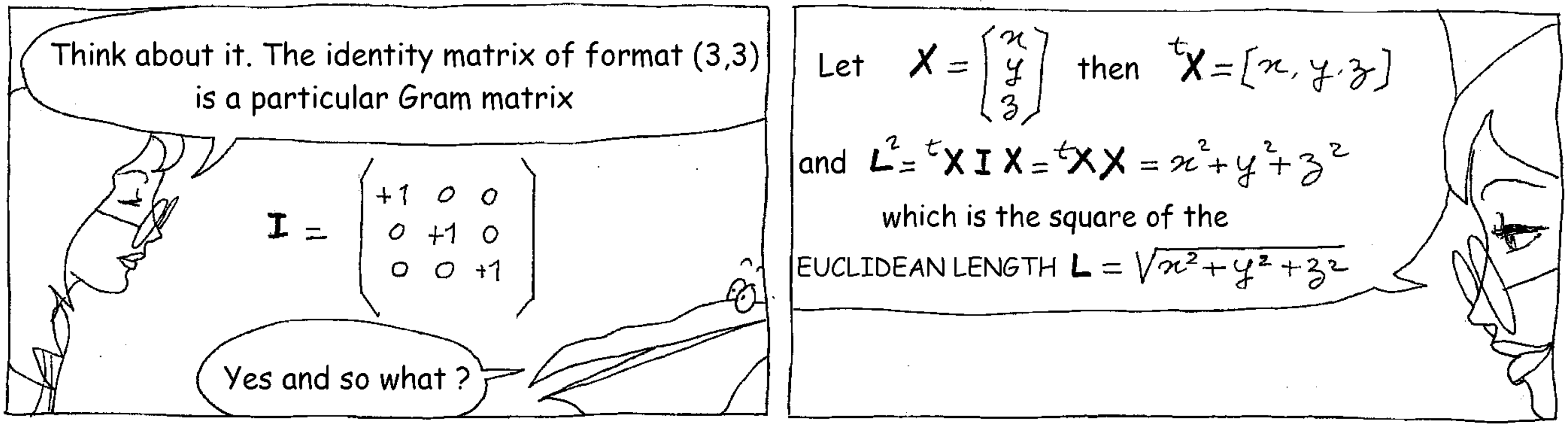
$$\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix} \begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix} \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix} \begin{pmatrix} \pm1 & 0 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 & 0 \\ 0 & 0 & \pm1 & 0 & 0 \\ 0 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 0 & \pm1 \end{pmatrix} \quad \text{etc...}$$

Let a vector X belonging to a space $\mathcal{E}$ with n dimensions. We will say that this space is Riemannian if the square of the length of the vector X is defined by :

$$\boxed{L^2 = {}^tX\, G\, X}$$



(*) Mathematicians are not all in agreement on the terminology. Here we decide to regroup under this name all spaces having a ±1 signature.



SIGNATURE

The signature of these spaces is the sequence of signs of the Gram metric. In the case of three dimensional euclidean space it's : $(+ + +)$

In a two dimensional space, the Gram matrix corresponding to an euclidean space would be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the signature $(+ +)$

We now ask the following question : is there a set of matrices M acting on the vector

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ and which preserve its length ?

We will now formally make the calculation in the most general case, which is the case of a Riemannian space with n dimensions defined by its Gram matrix G

Let M be a matrix acting on the vector X by transforming it into the vector:

$$X' = MX$$

The square of the length, of the norm of vector X' is

$$L'^2 = {}^tX' G X' = {}^t(MX)\, G\, (MX) = ({}^tX\, {}^tM)\, G' (MX) = {}^tX ({}^tM G M) X$$

the lengths L' and L will be equal if :

$$\boxed{{}^tM G M = G}$$

Let's apply this to an euclidean space of n dimensions :

$$\boxed{{}^tM M = I}$$

Which simply means that :

$$\boxed{M^{-1} = {}^tM}$$

Theses matrices are said to be orthogonal matrices. We will show it in the two dimensional case

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a^2 + b^2 = 1 \quad ; \quad c^2 + d^2 = 1 \quad ; \quad ac + bd = 0$$

We look for the matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which satisfy these relations.

These matrices M form a set ... $\mathcal{M}$

We will see that these matrices also form a ... **GROUP**

Here's the magical word of physics coming out ! But what is a group ?
It is a set of tricks which act on a set of things... In our case the tricks are MATRICES and the things are points, or set of points of a space.
Souriau has an habit to say:

- A group is made to transport.
- The manner that we transport is more important that what is being transported.

In the comic book we have read "tell me how you move, I will tell you WHAT you are".
Here we could say:

Tell me how you let yourself being transported and I will tell you in which family of geometrical beings you belong. In short, in which space you inhabit.

Hence the close relationship GROUP ⇄ GEOMETRY

The axioms that define a group were introduced by Norwegian Sophus Lie.
We also call group of matrices LIE GROUPS. Now let's look at the axioms:

Consider a set of things acting on each other. Let's call them α, β, γ ...
They form a set $\mathcal{E}$
We can compose them through a LAW OF COMPOSITION that we will write $\gamma = \alpha \circ \beta$

1: If α and β belongs to a set, $\alpha \circ \beta$ also belongs the set. We say that this law of composition is closed under group $\mathcal{E}$. (dogs do not make cats)

2: There exists an element e, called UNIT ELEMENT such as for all element a of the group, we have $\boxed{e \circ \alpha = \alpha \circ e = \alpha}$

3: Every element α has a INVERSE denoted α^{-1} such as :

$$\boxed{\alpha \circ \alpha^{-1} = e}$$

4: The composition operation is associative, meaning that:

$$\boxed{(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)}$$

we will almost NEVER use the fourth axiom. In fact it is very difficult to find operation of composition which are NOT ASSOCIATIVE.

The physicist will ONLY work on GROUP OF MATRICES also called LIE GROUP.

- We will have sets of square matrices M,
- The composition operation o will be the NON-COMMUTATIVE MATRIX PRODUCT $M_1 \times M_2$
- The unit element e will be systematically the identity matrix I in the considered format (n,n)

DISCRETE GROUPS

We call discrete groups those groups forming sets of finite elements. (here of matrices)
Gram matrices of format (2,2) form a group of 4 elements.

$$g = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \qquad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Incidentally, these matrices are identical to their inverse. What do they represent ?

Let them ACT on vectors $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ of a 2D space:

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ symmetry with respect to the oy axis.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ symmetry with respect to the ox axis.

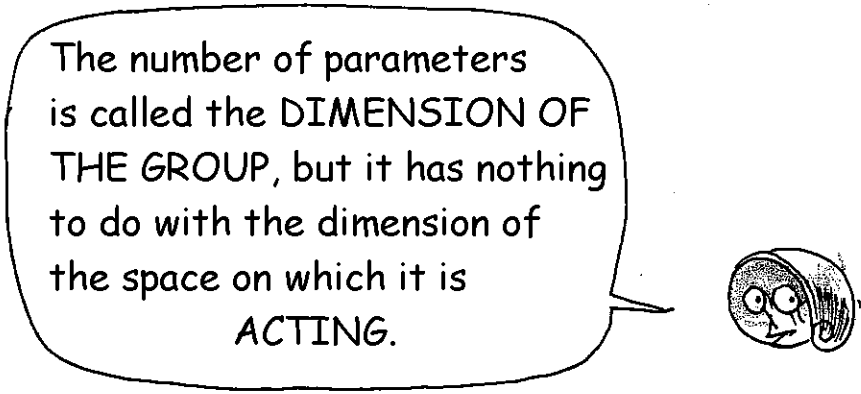
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ symmetry with respect to the origin.

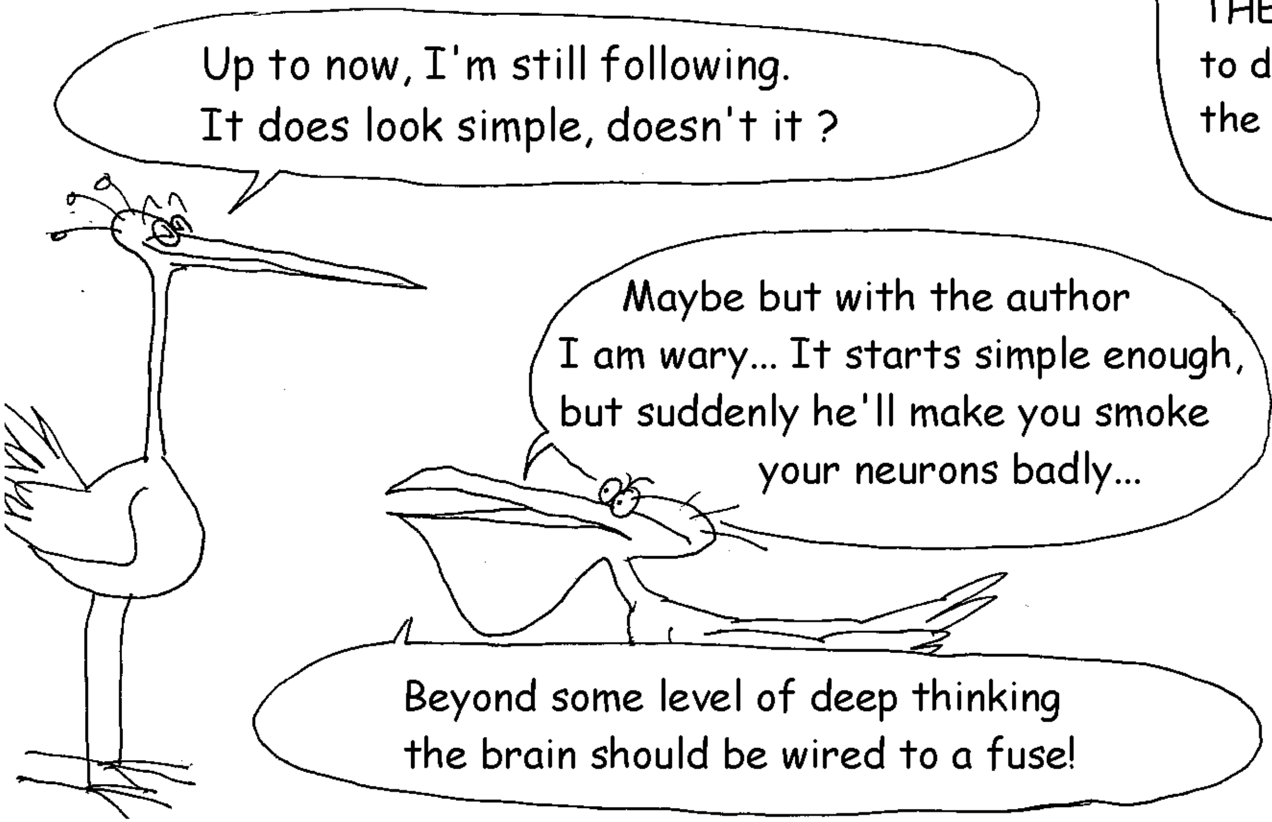
Our conditions are met: these symmetries conserve length.

GROUP WITH 1 (or many) PARAMETERS

The matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ satisfy our criteria and constitute the group of rotations of the plane around the origin.

It's a group with 1 parameter. (the angle θ)





The matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ form a group named SO(2), for "special orthogonal"

ORIENTATION

By multiplying this matrix by one of the two matrices inverting the objects (R ⇄ Я) for example the one which apply a symmetry with respect to the oy axis, we obtain:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

note that when $\theta = \pi$, we have a symmetry with respect to the ox axis.

We get a second set of matrices which are also orthogonal matrices since they satisfy ${}^{t}MM = I$. The union of these two sets constitute the ORTHOGONAL GROUP O(2). The element of this group will be denoted a and we will say that this group has TWO COMPONENTS.

$\boxed{O(2)}$ — $\boxed{SO(2)}$ → $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ constitutes a SUBGROUP of the group O(2) which does not inverse the objects : R → R

$\boxed{O(2)}$ → $\begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ does not constitute a group (the set does not contain the unit element). Its elements invert the objects: R ⇄ Я

ISOMETRY GROUP

The set of actions conserving lengths in a two dimensional space includes:

- **Rotations**
- **Symmetries**
- **Translations**

which can be expressed with matrices :

$\boxed{E(2)}$ $\boxed{SE(2)}$

$$\begin{pmatrix} \cos\theta & -\sin\theta & \Delta x \\ \sin\theta & \cos\theta & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \begin{pmatrix} x\cos\theta - y\sin\theta + \Delta x \\ x\sin\theta + y\cos\theta + \Delta y \\ 1 \end{pmatrix} \qquad \boxed{R \to R}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & \Delta x \\ \sin\theta & \cos\theta & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -x\cos\theta + y\sin\theta + \Delta y \\ x\sin\theta + y\cos\theta + \Delta y \\ 1 \end{pmatrix} \qquad \boxed{R \rightleftarrows Я}$$

We obtain the 2D EUCLIDEAN GROUP E(2) which is the ISOMETRY GROUP of the EUCLIDEAN SPACE in TWO DIMENSIONS. Its first COMPONENT SE(2) ("Special Euclid 2d") is a SUBGROUP. The second component is a set of matrices WHICH INVERT OBJECTS, but does not constitute a group.

In 2D it is possible to completely do the calculations explicitly. What has been done in 2D can be extended to 3D. The Gram matrix is the 3D identity matrix.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The square of the length is: $L^2 = {}^tX\, I\, X$ the signature: $(+\ +\ +)$

Let a matrix M acting on the vector X such that: $X = M X'$

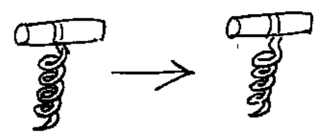
The conservation of the length leads to $L'^2 = {}^tX' I X' = {}^t(MX)(MX) = {}^tX({}^tMM)X$

$L' = L$ if:

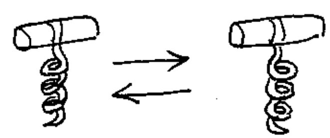
$$\boxed{{}^tMM = I \quad \text{ou} \quad M^{-1} = {}^tM}$$

The matrices having this property, which are square matrices (3,3), are said to be ORTHOGONALS and constitute the ORTHOGONAL GROUP O(3), which has TWO COMPONENTS:

O(3) → $\boxed{SO(3)}$ → Does not invert 3D objects



O(3) → Does invert 3D objects



(MIRROR SYMMETRY)

By adding the translation vector

$$\mathbf{c} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

We construct the 3D euclidean group E(3) which inherits from properties of the orthogonal group O(3) on top of which it is constructed. We will call a the element coming for O(3) and we will write :

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & a & & \Delta x \\ & (3,3) & & \Delta y \\ & & & \Delta z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \text{ acting on } \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

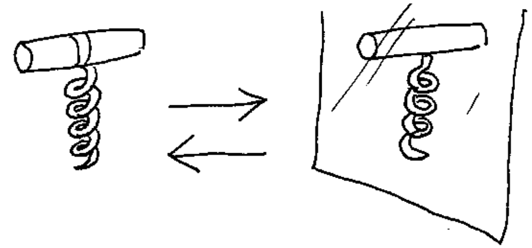
This ACTION, written in matrix form, allowing elements from the 3D euclidean group E(3) to act on the vector X, differs from the usual matrix multiplication of the like

$$X' = MX$$

which is just a form of ACTION among others. The concept of action is essential and we will reuse it later on.

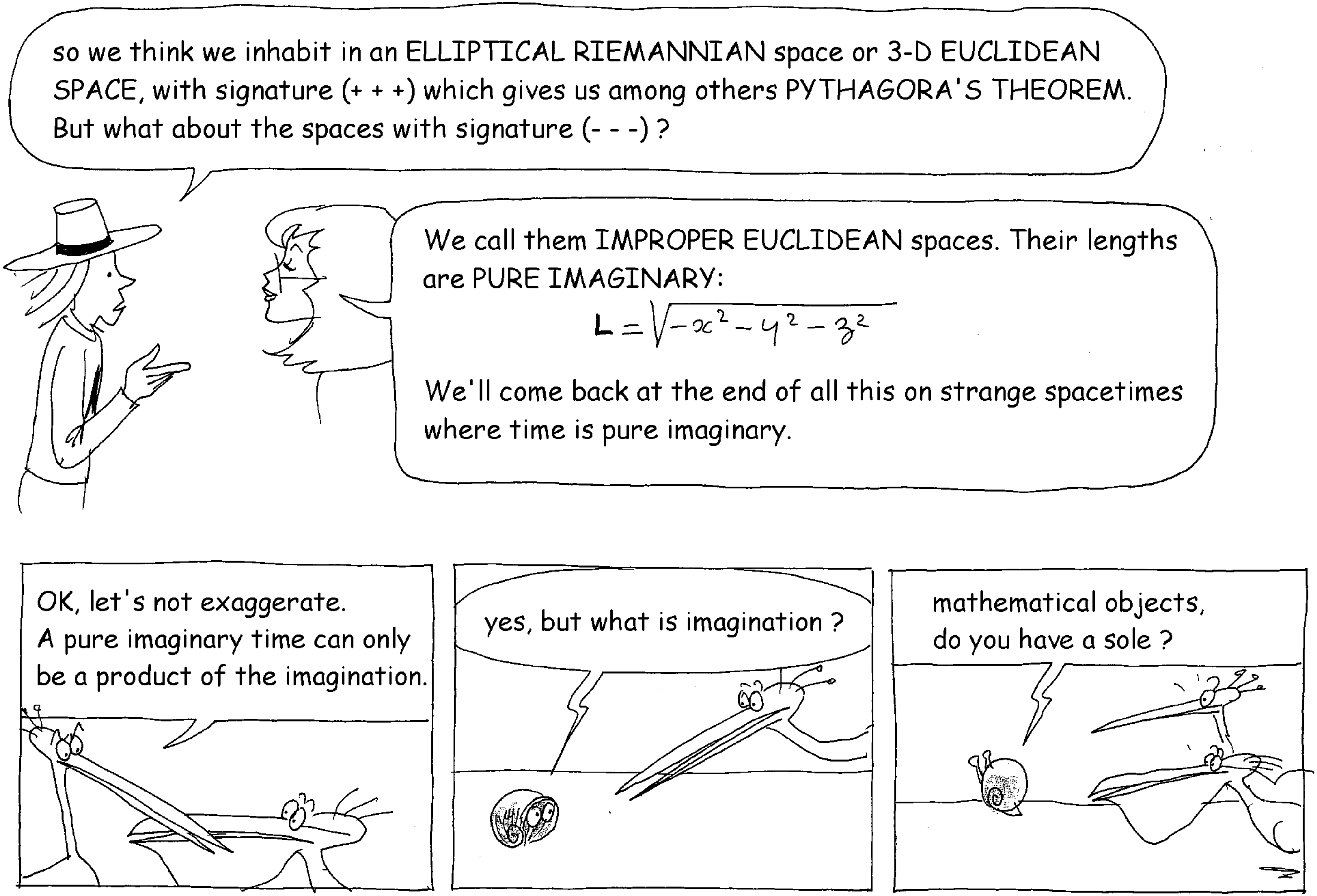
Half the matrices forming the euclidean group transform oriented objects (the cork-screw) in their mirror image. We will say that they operate

a P-SYMMETRY or "symmetry of parity".

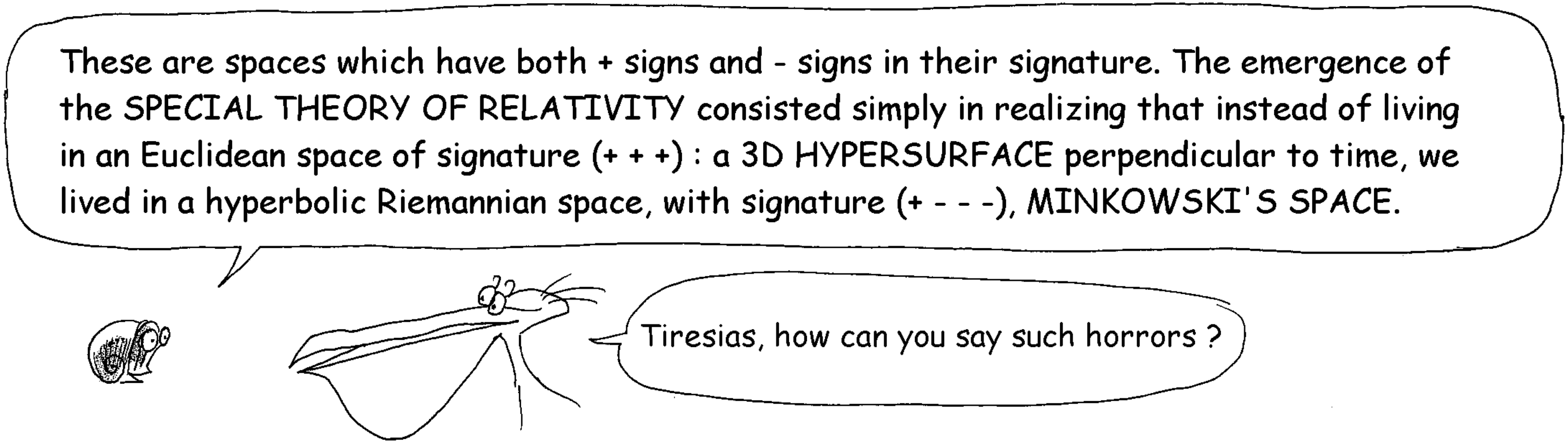


WHEN MATHEMATICIANS INVENT MIRRORS

Here for a few steps the mathematician precedes the physicist. After practicing rotations and translations, the mathematician invents the group notion, Gram matrices, construct the SE(3) subgroup which does not invert the objects by PHYSICALLY TRANSPORTING them. But the group produces elements that a simple physical transport could not create. By combining rotations and translations we will never create a "left-handed" cork-screw from a "right-handed" one. But the complete group predicts the "existence" of such ENANTIOMORPHIC objects living "on the other side of the mirror".


so we think we inhabit in an ELLIPTICAL RIEMANNIAN space or 3-D EUCLIDEAN SPACE, with signature (+ + +) which gives us among others PYTHAGORA'S THEOREM. But what about the spaces with signature (- - -) ?
We call them IMPROPER EUCLIDEAN spaces. Their lengths are PURE IMAGINARY:
$L = \sqrt{-x^2 - y^2 - z^2}$
We'll come back at the end of all this on strange spacetimes where time is pure imaginary.
OK, let's not exaggerate. A pure imaginary time can only be a product of the imagination.
yes, but what is imagination ?
mathematical objects, do you have a sole ?

HYPERBOLIC RIEMANNIAN SPACES



The GRAM matrix is then

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Let's change letter to designate a space-time vector :

$$\boldsymbol{\xi} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

We'll define a space-time translation vector which we'll write :

$$C = \Delta \boldsymbol{\xi} = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

We'll consider infinitesimal vectors :

$$d\,\boldsymbol{\xi} = \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

We will obtain (by taking c, the speed of light, = 1) the infinitesimal length :

$$\boxed{ds^2 = {}^t d\boldsymbol{\xi}\, \mathbf{G}\, d\boldsymbol{\xi} = dt^2 - dx^2 - dy^2 - dz^2}$$

which we'll call MINKOWSKI'S METRIC and we can write it with a simple change of variables:

$$\boxed{c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

We will proceed like we did for the euclidean group and the euclidean space. We will start by considering a 2D space-time:

$$\boldsymbol{\eta} = \begin{pmatrix} t \\ x \end{pmatrix}$$

where the element of length, its 2D metric is $\quad ds^2 = {}^t d\boldsymbol{\eta}\, \mathbf{G}\, d\boldsymbol{\eta}$
with, as Gram's metric :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We will construct the ISOMETRY GROUP of this space...

we will proceed like we did for the euclidean space. We'll set aside for a moment the presentation under the differential form. We are looking for a group of matrices L, acting on the vector ξ according to:

$$\xi' = L\xi$$

which preserves this strange "hyperbolic length", meaning such that:

$$L'^2 \quad {}^t\xi' G \xi' \quad {}^t(L\xi) G (L\xi) \quad {}^t\xi \left({}^tL G L\right) \xi = L^2 = {}^t\xi G \xi \quad \text{si :}$$

$$\boxed{{}^tL G L = G}$$

In 4D those are matrices with 4 lines, 4 columns (of format (4,4)). The above formula is the definition of the LORENTZ group (of matrices). To show it explicitly, we will limit ourselves to a 2D space-time (t,x)

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

giving $a^2 - c^2 = 1 \quad ; \quad b^2 - d^2 = 1 \quad ; \quad ab - cd = 0$

which gives us a first $\begin{bmatrix} ch\,\eta & sh\,\eta \\ sh\,\eta & ch\,\eta \end{bmatrix}$

since $\quad ch^2\eta - sh^2\eta = 1$

⇨ The trigonometric lines are replaced by hyperbolic lines

$$\begin{cases} \operatorname{ch}\eta = \dfrac{e^{\eta}+e^{-\eta}}{2} \\ \operatorname{sh}\eta = \dfrac{e^{\eta}-e^{-\eta}}{2} \end{cases} \qquad \begin{cases} \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} \end{cases} \qquad z = e^{i\theta} = \cos\theta + i\sin\theta$$

The LORENTZ GROUP is the equivalent of the rotations in MINKOWSKI's space.

DISCRETE GROUP

The 2D Gram matrices are Lorentz matrices, satisfying

$$ {}^{t}LGL = G $$

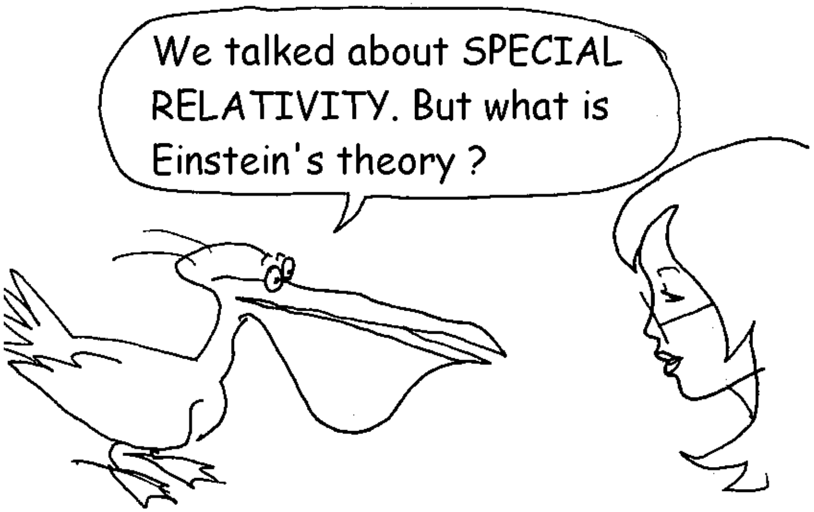
${}^{t}GGG = G$ with $GG = I$ and ${}^{t}G = G$, so in 2D we have the discrete group :

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

We will get the complete Lorentz group, with four components:

$$\underbrace{\boxed{\begin{pmatrix} \operatorname{ch}\eta & \operatorname{sh}\eta \\ \operatorname{sh}\eta & \operatorname{ch}\eta \end{pmatrix}} \; \boxed{\begin{pmatrix} \operatorname{ch}\eta & -\operatorname{sh}\eta \\ \operatorname{sh}\eta & -\operatorname{ch}\eta \end{pmatrix}}}_{\text{orthochron subgroup}} \; \underbrace{\boxed{\begin{pmatrix} -\operatorname{ch}\eta & \operatorname{sh}\eta \\ -\operatorname{sh}\eta & \operatorname{ch}\eta \end{pmatrix}} \; \boxed{\begin{pmatrix} -\operatorname{ch}\eta & -\operatorname{sh}\eta \\ -\operatorname{sh}\eta & -\operatorname{ch}\eta \end{pmatrix}}}_{\text{antichron subset}}$$

SPECIAL RELATIVITY

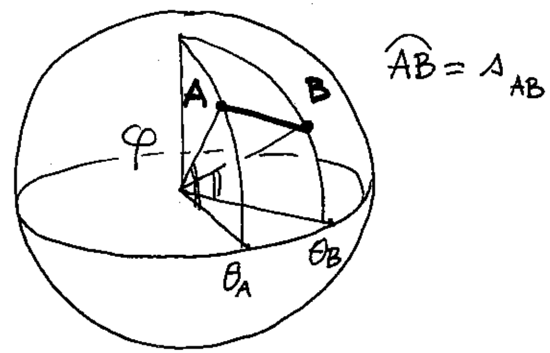


go back to the length's calculation in this hyperbolic Riemann space that is the MINKOWSKI'S SPACE in differential form, given by the metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This means that our MOVEMENTS ARE WRITTEN (*) on a 4D hypersurface. (x,y,z,t) are COORDINATES on it. In FASTER THAN LIGHT we explain that the inscribing of a coordinates system on this hypersurface corresponds to the observation done by the PHYSICIST of this hypersurface where the only INTRINSIC value is the length S. There is the same relation between these coordinates and this length S, which is measured in METERS and which is converted in PROPER TIME τ using the relation ds = cdt where c is the characteristic speed only between the coordinates of longitude θ and latitude φ used for finding points on a sphere and the length of the trajectory $\widehat{AB}$. What is shown by this formula is that when we take coordinates (x,y,z,t), we can deduce a speed

$$V = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$$



For the time $d\tau$ to remain real , we must have $V < c$

the limit movement will correspond to $V = c$, and then $d\tau = 0$

⇨ the proper time of the PHOTON is "frozen"

(*) in Arabic: MEKTOUB

For particles traveling at $V<c$ we have LORENTZ'S CONTRACTION applying

$$\boxed{c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \Rightarrow \frac{d\tau}{dt} = \sqrt{1 - \frac{V^2}{c^2}}}$$

τ is the time showing on the traveller's watch moving at velocity V, which is illustrated in the album EVERYTHING IS RELATIVE. And when V approaches c "time is freezing in the chronometers". But let's come back to LORENTZ'S GROUP. Its elements act on a series of points of spacetime which constitute a MOVEMENT. By letting an element L of the Lorentz group on a given movement we obtain another movement. The fact the group contains ANTICHRONS elements shows that these TIME-REVERSED movements have to be taken into consideration. For example, here's a matrix which belongs to the Lorentz group:

$$L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^tL\,G\,L = G \quad \text{with} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The action is:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ x \\ y \\ z \end{pmatrix}$$

TIME INVERSION

When we defined the ORTHOGONAL GROUP, subgroup of the isometry group of EUCLIDEAN SPACE, we completed it with the SPATIAL TRANSLATIONS vector

$$\mathbf{c} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

by constructing the EUCLIDEAN GROUP, its isometry group

element of orthogonal group O(3)

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Similarly, from LORENTZ'S GROUP we are going to construct the POINCARÉ GROUP, the isometry group of MINKOWSKI'S SPACE.

$$C = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$ spacetime translations

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \xi \\ 1 \end{pmatrix} \qquad \xi = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

The Poincaré group, through its subgroup $\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}$ inherits of the properties of the Lorentz group and likewise has four components:

- TWO ORTHOCHRONS (not inverting time)
- TWO ANTICHRONS (inverting time)

We still have to understand the PHYSICAL SIGNIFICANCE of this temporal inversion

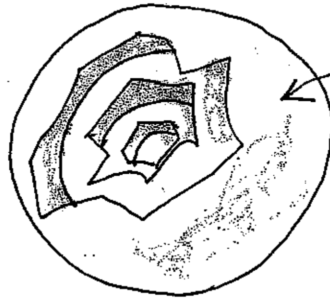
SPACE, GROUPS AND OBJECTS

We started from the euclidean space and limited ourselves to 2D so that we could show the calculations explicitly. We then construct its ISOMETRY GROUP, the EUCLIDEAN GROUP. This group goes along the euclidean space and can ACT on the objects, points living in this space. But we can take the problem backward: take a group, as an abstract object, purely mathematical, allowing to envisions ACTIONS and discover the "space that goes along", the only one where these actions can be realized - "the matching space" in other words. Hence space and its (isometry) group mutually gives themselves their existences.

But there is more - the group generate the OBJECTS of the space to which it is linked by the INVARIANCES OF THE ACTIONS OF A SUBGROUP. Let's give an example: the rotations around a point in 2D euclidean space constitute one of its subgroups. The invariants objects are then the family of circles centered on this point. This is how, in terms of groups, that we define the circle!

Lucretius, poet and roman philosopher, 1st century BC, imagined that the objects were made of atoms by comparing the analogy between the flow of water and of sand (See FLIGHT OF FANCY pages 15 to 17)

In the 3D euclidean group, the rotations around a point also constitute one of its subgroups. What are the objects that these ACTIONS OF THE SUBGROUP leaves INVARIANT ? Answer: the family of SPHERES centered on that point. The concept of INVARIANT by such or such action of the group or one of its subgroups is a fundamental concept of GROUP THEORY.



$$\begin{bmatrix} a & c \\ 0 & 1 \end{bmatrix}$$

In the euclidean group, where time is absent, the group generates itself the OBJECTS which will populate the space to which it is associated.

When time enters the picture, the group becomes a DYNAMIC GROUP. It no longer manages static objects, but SET OF "EVENT-POINTS" that we can name TRAJECTORIES or MOVEMENTS. At the beginning of the 20th century, the remarkable German mathematician Emmy Noether (qualified by Einstein as "movement of physics") gave her name to one of the most important theorem of physics that says that for every subgroup of a dynamic group corresponds an INVARIANT.

In the POINCARÉ GROUP we find the SUBGROUP OF TIME TRANSLATIONS, represented by the matrix on the right. Group with 1 parameter, there is a corresponding invariant, a scalar: the ENERGY E This is how, in terms of groups, that we define energy!

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \Delta t \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} t+\Delta t \\ x \\ y \\ z \\ 1 \end{pmatrix}$$

Second subgroup: the subgroup of SPATIAL TRANSLATIONS (matrix on the right), group with three parameters (Δx, Δy, Δz).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta x \\ 0 & 0 & 1 & 0 & \Delta y \\ 0 & 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ x+\Delta x \\ y+\Delta y \\ z+\Delta z \\ 1 \end{pmatrix}$$

A new invariant corresponds to this subgroup:

the MOMENTUM $\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$

this is how, with the help of DYNAMIC GROUPS that we define momentum. In that way, the quantifiable values of physics become GEOMETRICAL OBJECTS and this process of GEOMETRISATION OF PHYSICS constitute of the the pillars of MODERN PHYSICS.

By continuing playing that little game we could consider the subgroup of SPACETIME TRANSLATIONS (matrix on the right)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \Delta t \\ 0 & 1 & 0 & 0 & \Delta x \\ 0 & 0 & 1 & 0 & \Delta y \\ 0 & 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} t+\Delta t \\ x+\Delta x \\ y+\Delta y \\ z+\Delta z \\ 1 \end{pmatrix}$$

The invariant object would then be the MOMENTUM-ENERGY FOUR-VECTOR. $\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$

What is the use of QUANTIFIABLE VALUES IN PHYSICS ? Good question.
Answer = WE CAN ADD THEM UP !

The Poincaré group depends on ten parameters (we say that it has "ten dimensions" in simple math nerd terminology). There are 3 for spatial translations, 1 for temporal dimension. There remains six, which represent the dimension of the LORENTZ GROUP, which manage "spacetime rotations". If we consider the Lorentz group as a subgroup of the Poincaré group :

$$\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \xi \\ 1 \end{pmatrix} = \begin{pmatrix} L\xi \\ 1 \end{pmatrix} \text{ with } \xi = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Noether's theorem says that it must have a corresponding "object" defined by six parameters which will be invariant under the action of this subgroup.

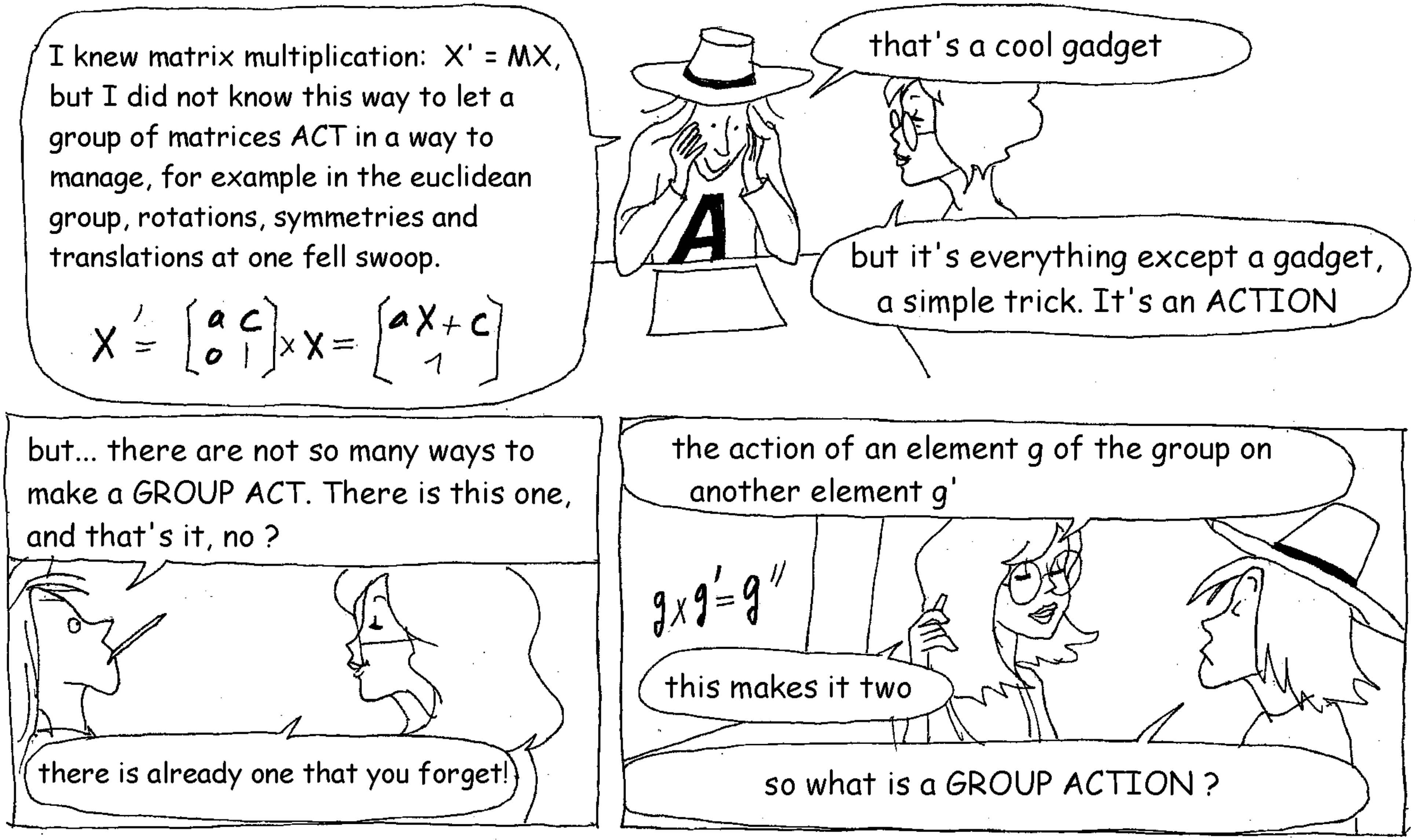
In this object, SPIN is hiding. Souriau showed in 1972 its PURELY GEOMETRIC nature. It has the dimension of angular momentum. Now the Poincaré group manages the movements of the RELATIVISTIC MATERIAL POINT. The interpretation of spin as a purely geometric object is preferable.

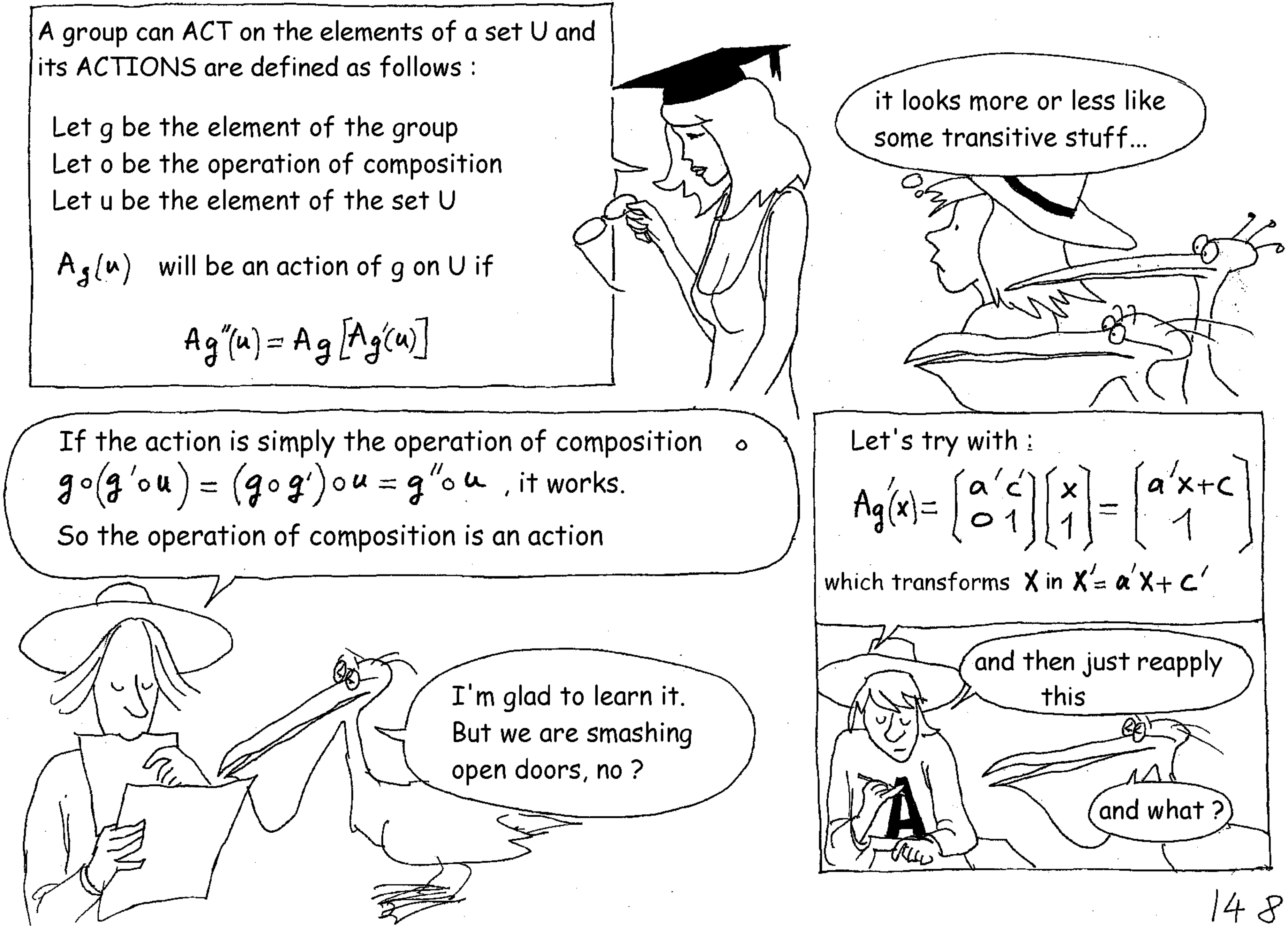
The "MOMENT"

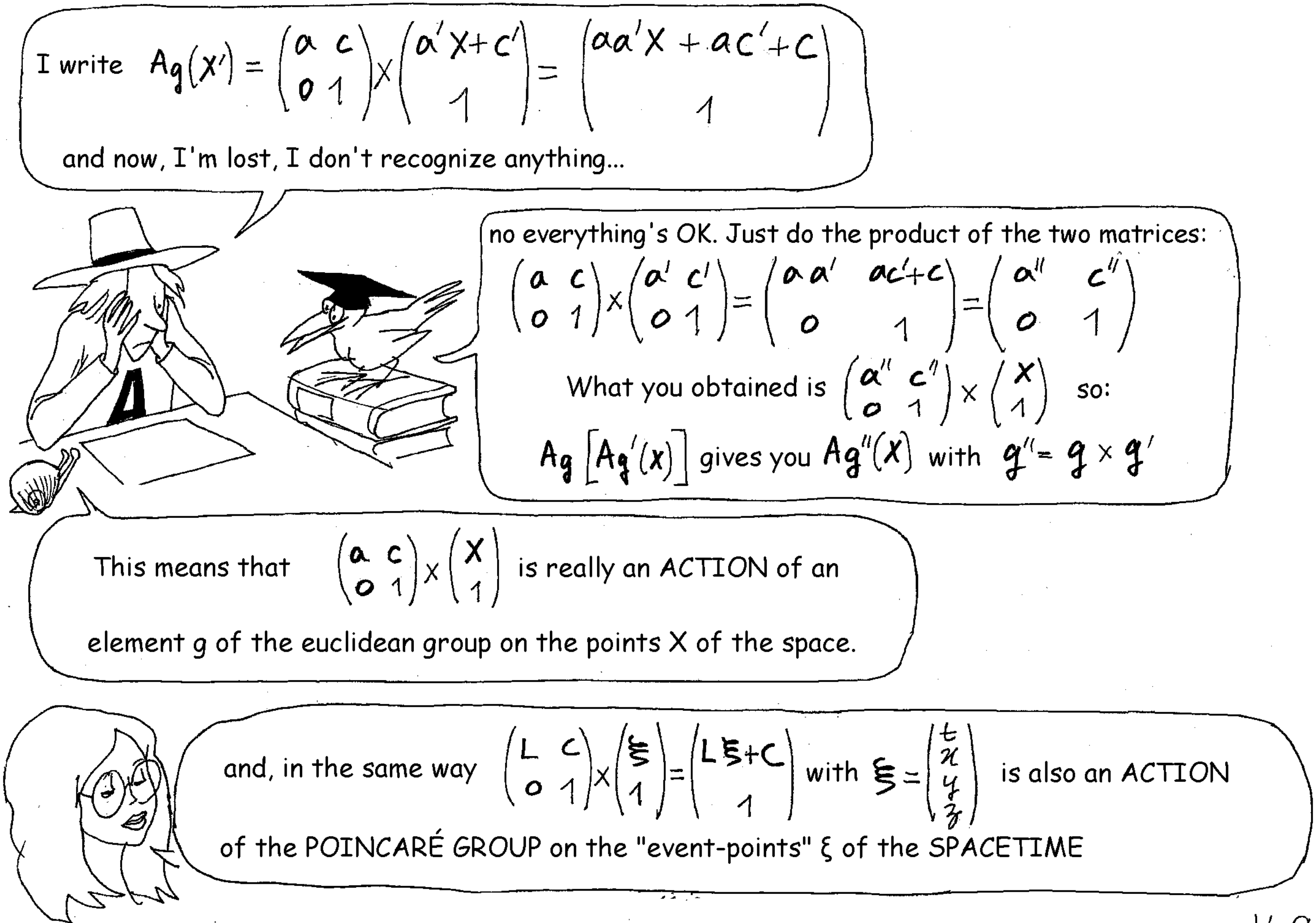
The subgroups correspond to a kind of "dismantling of the group, part by part". When we do the opposite operation, we reconstitute the group. The set of invariants found earlier constitute what Souriau has called the "moment"

$$\textbf{moment} = \{ E, p_x, p_y, p_z, \cdots \text{SPIN} \}$$

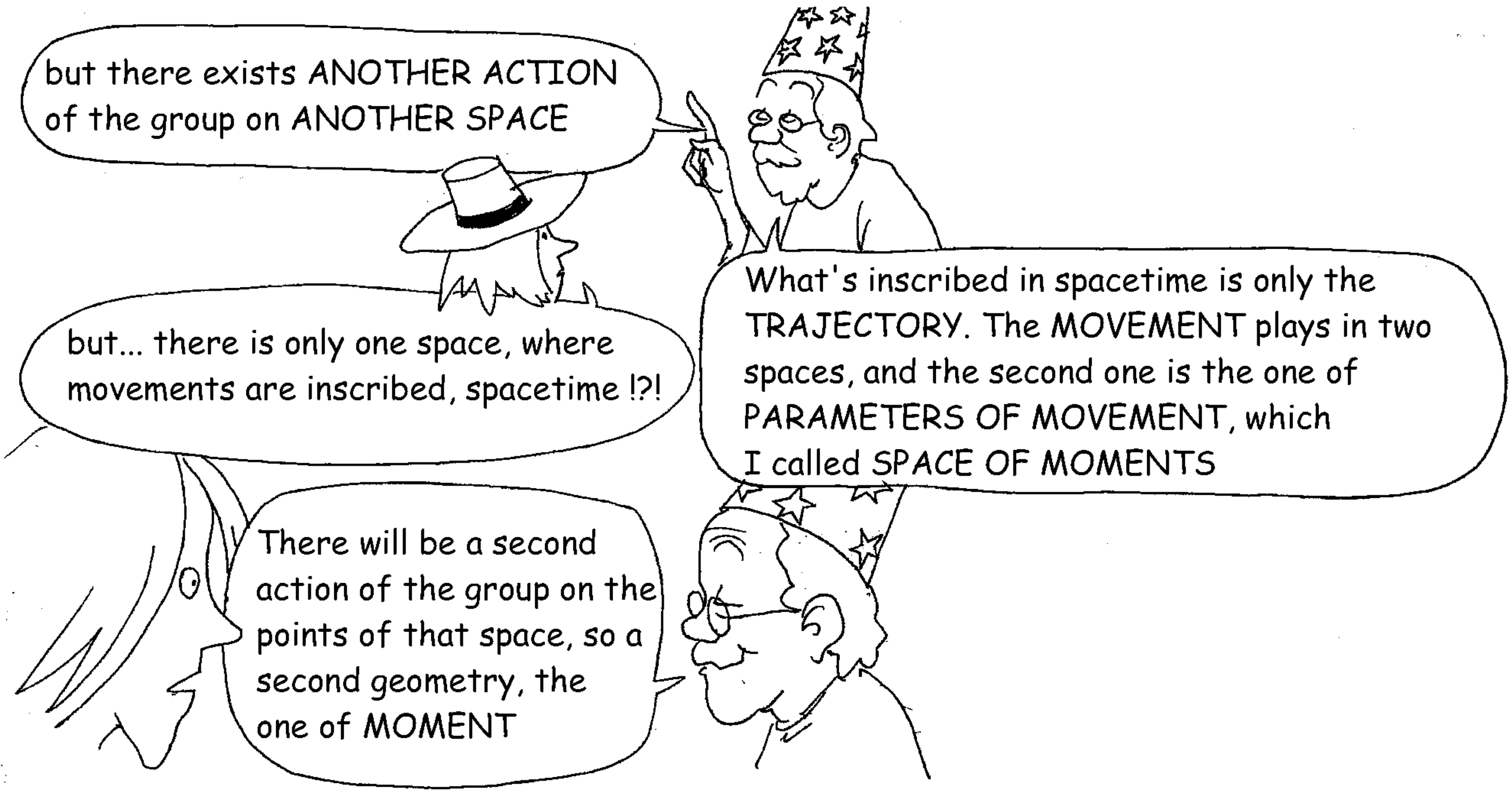
ACTIONS OF A GROUP




A group can ACT on the elements of a set U and its ACTIONS are defined as follows :
Let g be the element of the group
Let o be the operation of composition
Let u be the element of the set U
$A_g(u)$ will be an action of g on U if
$Ag''(u) = Ag[Ag'(u)]$
it looks more or less like some transitive stuff...
If the action is simply the operation of composition $\circ$
$g \circ (g' \circ u) = (g \circ g') \circ u = g'' \circ u$, it works.
So the operation of composition is an action
I'm glad to learn it. But we are smashing open doors, no ?
Let's try with :
$A_g'(x) = \begin{bmatrix} a' & c' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} a'x + c \\ 1 \end{bmatrix}$
which transforms X in $X' = a'X + c'$
and then just reapply this
and what ?
A


I write $A_g(X') = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a'X + c' \\ 1 \end{pmatrix} = \begin{pmatrix} aa'X + ac' + c \\ 1 \end{pmatrix}$
and now, I'm lost, I don't recognize anything...
no everything's OK. Just do the product of the two matrices:
$\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a' & c' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ac'+c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a'' & c'' \\ 0 & 1 \end{pmatrix}$
What you obtained is $\begin{pmatrix} a'' & c'' \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} X \\ 1 \end{pmatrix}$ so:
$A_g\left[A_{g'}(X)\right]$ gives you $A_{g''}(X)$ with $g'' = g \times g'$
This means that $\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} X \\ 1 \end{pmatrix}$ is really an ACTION of an element g of the euclidean group on the points X of the space.
and, in the same way $\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \xi \\ 1 \end{pmatrix} = \begin{pmatrix} L\xi + C \\ 1 \end{pmatrix}$ with $\xi = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ is also an ACTION of the POINCARÉ GROUP on the "event-points" ξ of the SPACETIME

BEWARE A GEOMETRY CAN HIDE ANOTHER!



$$\boxed{J' = g \times J \times {}^t g}$$ where J is an ANTISYMMETRIC matrix

we can verify that it is indeed an ACTION

$$A_g[A_{g'}(J)] = g \times [g' \times J \times {}^t g'] \times {}^t g = g\, g'\, J\, {}^t g'\, g$$

but ${}^t[AB] = {}^tB\,{}^tA$ then ${}^t g'\,{}^t g = {}^t(g\,g')$ and if $g'' = g\,g'$

$$A_g[A_{g'}(J)] = g'' \quad {}^t g'' = A_{g''}(J)$$

The J matrix necessarily has the same format (5,5) of the g matrices of the group. In an antisymmetric matrix, the symmetric elements with respect to the main diagonal have opposite signs. The elements of the main diagonal are equal to zero (which is its own opposite). We can now count the components of this matrix

$$\begin{pmatrix} 0 & l \\ -l & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -l_z & -l_y \\ l_z & 0 & -l_x \\ -l_y & l_x & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -l_z & l_y & f_x \\ l_z & 0 & -l_x & f_y \\ -l_y & l_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -l_z & l_y & f_x & -p_x \\ l_z & 0 & -l_x & f_y & -p_y \\ -l_y & l_x & 0 & f_z & -p_z \\ -f_x & -f_y & -f_z & 0 & -E \\ p_x & p_y & p_z & E & 0 \end{pmatrix}$$

(2,2) (3,3) (4,4) (5,5)

Format	Number of components
(2,2)	1
(3,3)	3
(4,4)	6
(5,5)	10

I can decompose this antisymmetric matrix J of format (5,5) in an antisymmetric matrix M of format (4,4) and a FOUR-VECTOR p, with four components. And I will be able to write this in a more compact manner. Quite simply, this will allow me to show the calculation of the action of the Poincaré group on this moment-matrix J in a more convenient way.

$$J = \begin{pmatrix} 0 & -l_z & l_y & f_x & -p_x \\ l_z & 0 & -l_x & f_y & -p_y \\ -l_y & l_x & 0 & f_z & -p_z \\ -f_x & -f_y & -f_z & 0 & -E \\ p_x & p_y & p_z & E & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -l_z & l_y & f_x \\ l_z & 0 & -l_x & f_y \\ -l_y & l_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} \begin{pmatrix} -p_x \\ -p_y \\ -p_z \\ -E \end{pmatrix} \quad \begin{pmatrix} p_x & p_y & p_z & E \end{pmatrix} \quad \begin{pmatrix} 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & -l_z & l_y & f_x \\ l_z & 0 & -l_x & f_y \\ -l_y & l_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} \qquad P = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E \end{pmatrix}$$

$${}^tP = \begin{pmatrix} p_x & p_y & p_z & E \end{pmatrix}$$

$$J = \begin{pmatrix} M & -P \\ {}^tP & 0 \end{pmatrix} \qquad g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

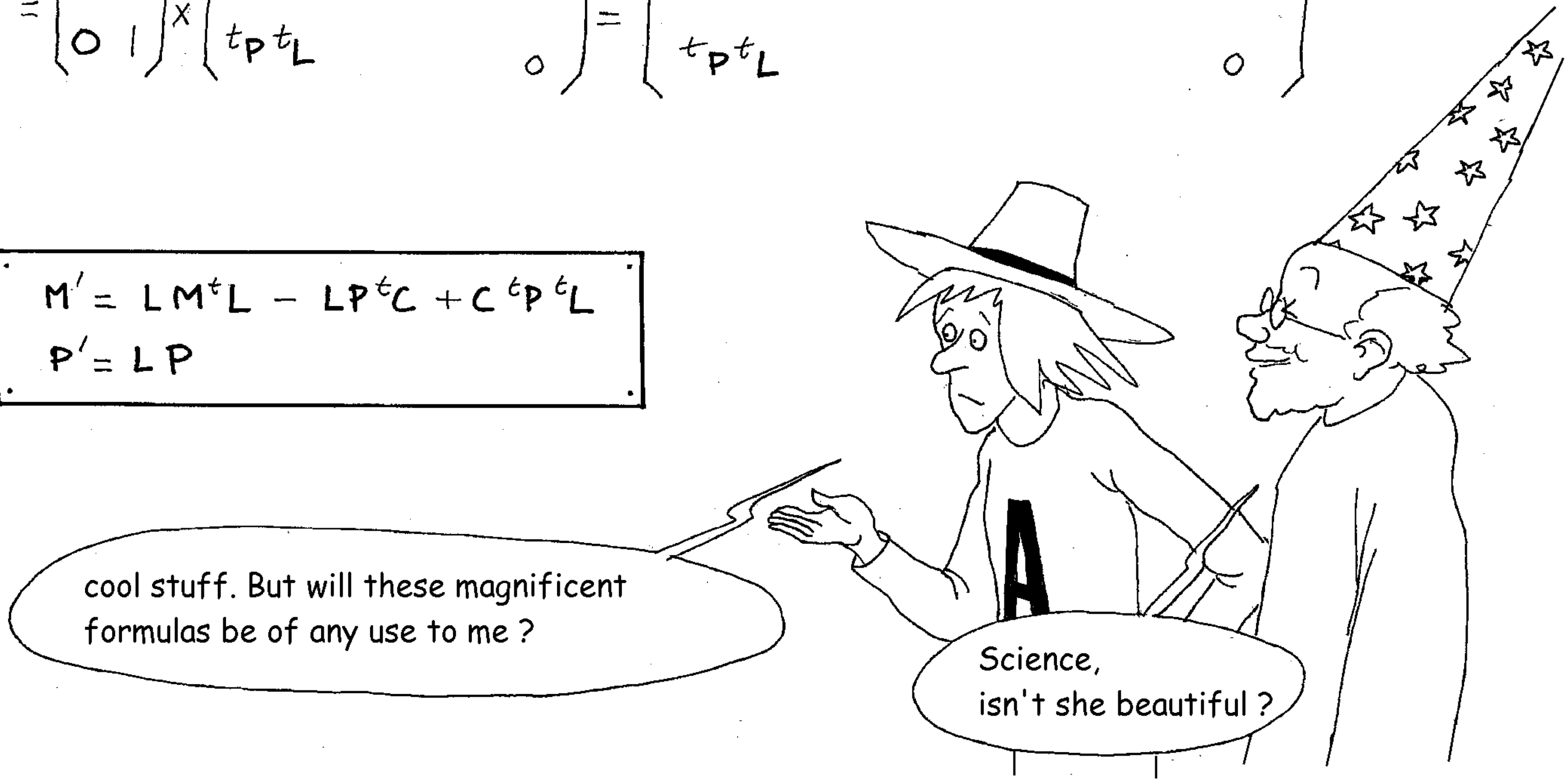
from that point of view, this decomposition is logical

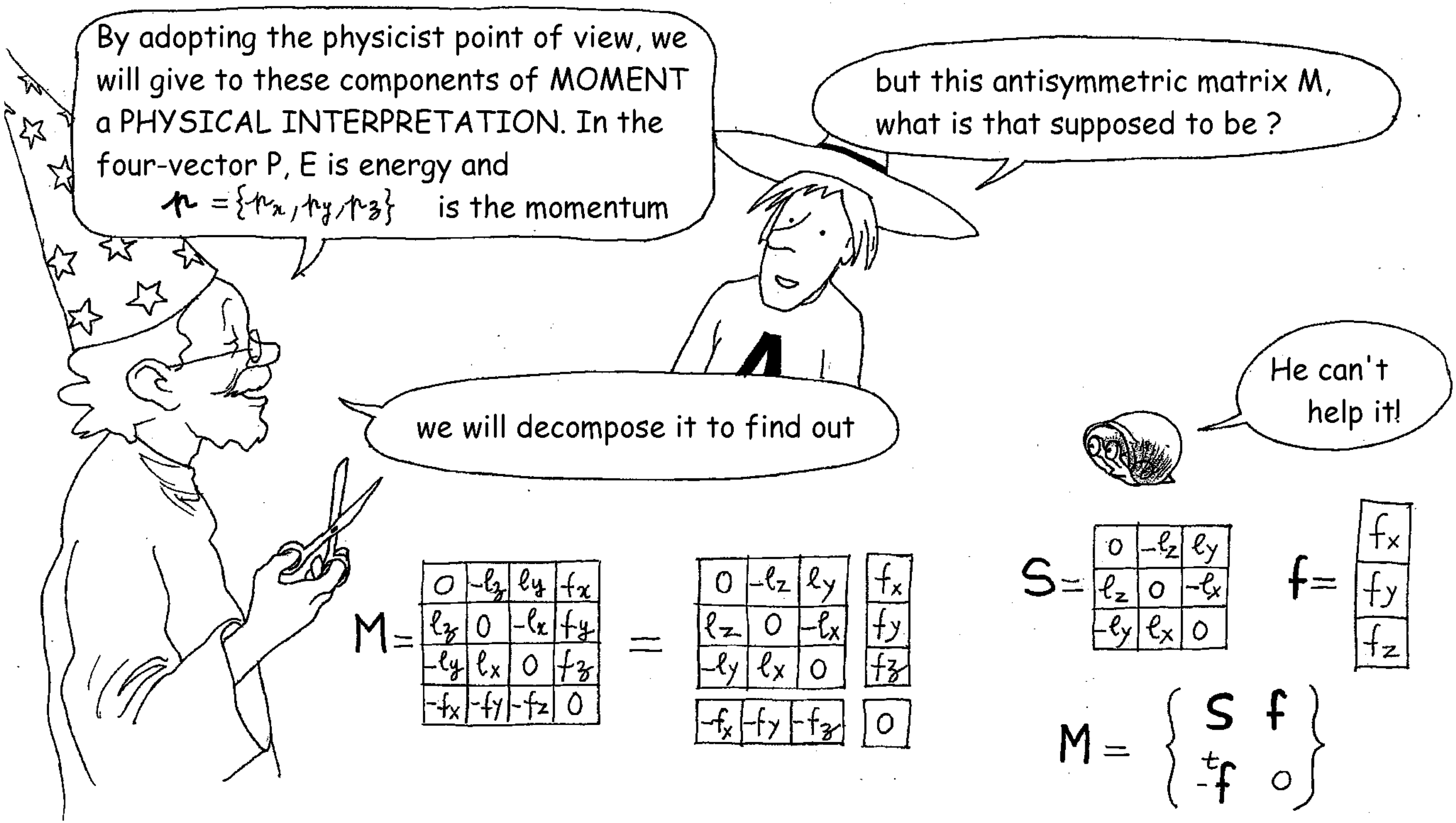
we just have to show the details of $J' = g \times J \times {}^tg$

$$ {}^t g = \begin{pmatrix} {}^tL & 0 \\ {}^tC & 1 \end{pmatrix} \qquad J' = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} M & -P \\ {}^tP & 0 \end{pmatrix} \times \begin{pmatrix} {}^tL & 0 \\ {}^tC & 1 \end{pmatrix} $$

$$ J' = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} M\,{}^tL - P\,{}^tC & -P \\ {}^tP\,{}^tL & 0 \end{pmatrix} = \begin{pmatrix} L M\,{}^tL - L P\,{}^tC + C\,{}^tP\,{}^tL & -LP \\ {}^tP\,{}^tL & 0 \end{pmatrix} $$

$$ \boxed{\begin{aligned} M' &= L M\,{}^tL - L P\,{}^tC + C\,{}^tP\,{}^tL \\ P' &= L P \end{aligned}} $$





$$M = \begin{pmatrix} 0 & -\ell_z & \ell_y & f_x \\ \ell_z & 0 & -\ell_x & f_y \\ -\ell_y & \ell_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 0 & -\ell_z & \ell_y \\ \ell_z & 0 & -\ell_x \\ -\ell_y & \ell_x & 0 \end{pmatrix} & \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\ \begin{pmatrix} -f_x & -f_y & -f_z \end{pmatrix} & \begin{pmatrix} 0 \end{pmatrix} \end{array}$$

$$S = \begin{pmatrix} 0 & -\ell_z & \ell_y \\ \ell_z & 0 & -\ell_x \\ -\ell_y & \ell_x & 0 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$M = \begin{Bmatrix} S & \mathbf{f} \\ -{}^t\mathbf{f} & 0 \end{Bmatrix}$$

The velocity V is implicitly present in the L matrix of the Lorentz group. If we consider a movement which takes place along a specific direction, for example oz with a velocity V and a translation $\Delta z = c$ and if $c = V\,\Delta t$ then we are in a system of coordinates where we follow the particle's movement along this spacetime translation. We then show that the vector f is null.

The matrix S is then written:

$$\begin{array}{|c|c|c|} \hline 0 & -s & 0 \\ \hline s & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

The GEOMETRIC QUANTIFICATION method that he invented allows to show that this spin S can only be a multiple of a fixed quantity: ħ. We have seen that the fact that a particle has an electric charge was equivalent of saying that it moves in a space having a FIFTH DIMENSION, the dimension of KALUZA. It's the fact that this dimension is closed onto itself that causes the electric charge to be quantized. In spacetime, there exists a "form of closure" that cause an object to become identical to itself under the action of a 360° rotation. The quantization of Spin, in a certain measure, comes from that property. There exists a close relationship between quantization and closure of a dimension. By exploiting the "group" tool and closure of the 5th dimension, Souriau shows the emergence the Klein-Gordon equation of the Poincaré group (and the Schrödinger equation of the Galilean group, dynamic group managing movement of the non-relativistic material point)

INVERSION OF ENERGY FOLLOWS FROM INVERSION OF TIME

We've seen earlier that the element from Lorentz group could be written in the form:

$$L = \mu L_o \qquad \mu = \pm 1$$

where L_0 represents the element of the orthochron subgroup (with does not invert time). In this form the action is written:

$$M' = L_o M \, {}^tL_o - \mu L_o P \, {}^tC + \mu C \, {}^tP L_o$$

$$P' = \mu L_o P$$

Let's consider the simplest action possible where there is time inversion ($\mu = -1$). In the orthochron L_0, let's choose the identity matrix I. Let's cancel the spacetime translation C. The element of the group is written:

$$g = \begin{pmatrix} -I & 0 \\ 0 & 1 \end{pmatrix}$$

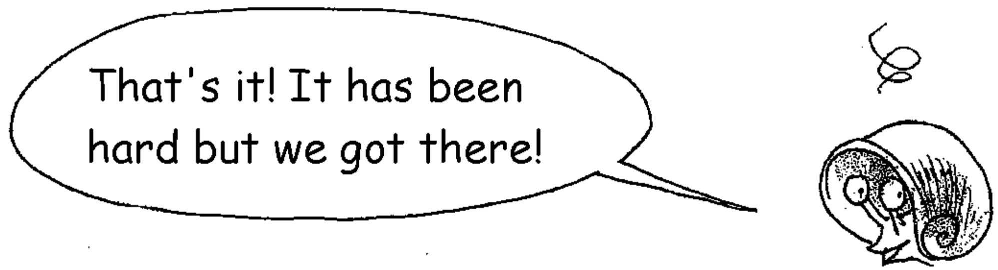
The action on spacetime, the space of trajectories is reduced to:

$$\xi' = -\xi \quad \Rightarrow \quad t \Rightarrow -t$$

It's the inversion of the direction of time along the trajectory. The action on the moment is:

$M' = M \quad \Rightarrow$ the spin S remains unchanged.

$$P' = -P \quad : \quad E \rightarrow -E$$



APPENDIX 4: THE ANTIMATTER

On page 40 we evoked the idea that for a relativistic material point to have an electric charge e, we must consider its displacement not in a four dimensional space, but in a space of five dimensions:

$$\{t, x, y, z, \zeta\}$$

ζ being the fifth dimension, or KALUZA'S DIMENSION.
We had introduced MINKOWSKI'S METRIC on page 137

$$\boxed{ds^2 = {}^t d\boldsymbol{\xi}\, G\, d\boldsymbol{\xi} = dt^2 - dx^2 - dy^2 - dz^2}$$

we will start from a KALUZA SPACE, hyperbolic Riemannian, defined by its signature (+ - - - -) and its Gram matrix :

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & -1 \end{pmatrix} \text{ where } G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The metric of the Kaluza space is :

$$d\Sigma^2 = dt^2 - dx^2 - dy^2 - dz^2 - d\zeta^2$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \boldsymbol{\xi} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} t \\ x \\ y \\ z \\ \zeta \end{pmatrix} = \begin{bmatrix} \boldsymbol{\xi} \\ \zeta \end{bmatrix} = \begin{pmatrix} t \\ \mathbf{r} \\ \zeta \end{pmatrix}$$

$$d\Sigma^2 = {}^t d\boldsymbol{\Omega}\, \Gamma\, d\boldsymbol{\Omega}$$

if we look for the isometry group of this Kaluza space we will find a group which matrix representation looks very much like the one from a Poincaré group but with an extra dimension:

$$\begin{pmatrix} \Lambda & C \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad {}^t\Lambda \Gamma \Lambda = \Gamma$$

this group acts on the points in the Kaluza space:

$$\begin{pmatrix} \Lambda & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{\Omega} \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda \boldsymbol{\Omega} + C \\ 1 \end{pmatrix}$$

The vector C represents this time a translation with five dimensions:

$$C = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \\ \Delta \zeta \end{pmatrix}$$

the translations along dimension ζ represents a subgroup of this group:

of which the matrix representation is :

subgroup with 1 parameter

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \Delta\zeta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} t \\ x \\ y \\ z \\ \zeta \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \\ \zeta + \Delta\zeta \\ 1 \end{pmatrix}$$

Now Noether's theorem says that a new scalar will be invariant under the action of this subgroup, and this scalar is

$$\boxed{\textbf{THE ELECTRIC CHARGE} \quad e}$$

the Kaluza group is constructed from a group Λ

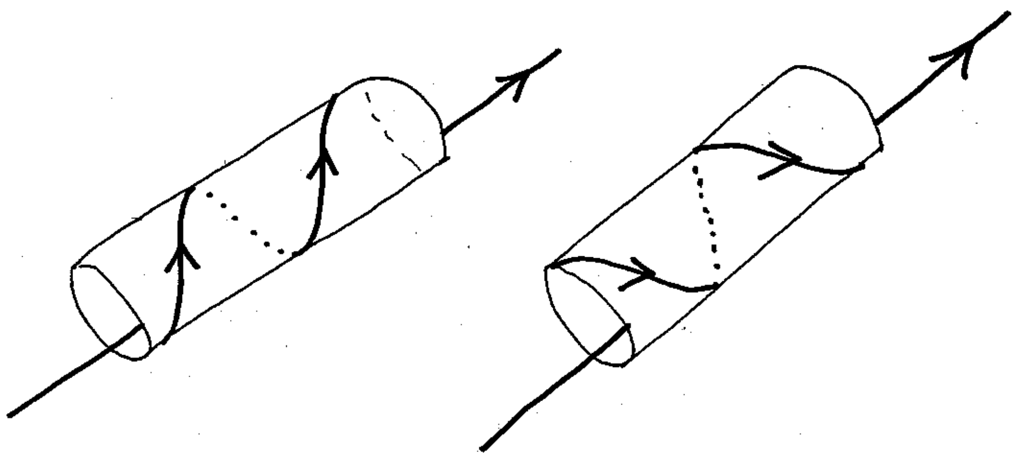
the Lorentz group is one of its subgroups:

$$\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}$$

here's another subgroup from the Kaluza group

$$\begin{bmatrix} L & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \xi \\ \zeta \\ 1 \end{bmatrix} = \begin{bmatrix} L\xi \\ \mu\zeta \\ 1 \end{bmatrix} \quad \text{with} \quad \mu = \pm 1$$

the elements ($\mu = -1$) of this group invert the fifth dimension. We reuse the sketch from page 42 : (the fifth dimension is closed)



The "wrapping direction" of the movement of the particle is reversed. We show (...) that this involves the inversion of the electrical charge **e**

This cannot represent a geometrical definition of antimatter. A particle has QUANTUM CHARGES and the electric charge **e** is only one of them. But we can see the idea coming up: "the antimatter statute depends of a type of movement in a space of higher dimension"

ORTHOCHRON and ANTICHRON LORENTZ SUBGROUP

The LORENTZ GROUP L has four components

L_n (neutral), L_s (inverts space), L_t (inverts time), L_{st} (inverts space and time)

The "neutral component" is a subgroup which contains the unit element, unlike the three other sets and does not inverts neither time or space. Below, a few matrices which belongs to the sets ($\in$ means "belongs to" and { } means set)

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \in \{L_n\} \; ; \; \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix} \in \{L_s\} \; ; \; \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \in \{L_t\} \; ; \; \begin{pmatrix} -1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix} \in \{L_{st}\}$$

APPENDIX 5
TWIN GROUP

We can regroup these four sets of matrices in two subsets :

$$L_o \text{ (orthochron)} = \{ L_n , L_s \} \qquad L_a = \{ L_t ; L_{st} \}$$

The first subset is a subgroup of the Lorentz group. This regrouping allows us to write :

$$L = \mu L_o \quad \text{with} \quad \mu = \pm 1 \text{ because } L_t = - L_s \quad ; \quad L_{st} = - L_n$$

In this large matrix calculation that we didn't dare putting on these pages (but that you could easily follow), the most general "ACTION" of the components of the Poincaré group on its "moments space" contains the relation (Souriau 1972)

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = L \times \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \mu L_o \times \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

The elements μ = -1 correspond to the ANTICHRON transformations which invert time. The identity matrix (4,4) I is part of the Lorentz group. When we limit ourselves to just inverting time, we see that it inverts the energy, but also the momentum p

$$p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\boxed{E' = -E \qquad p' = -p}$$

If we take the Kaluza group

$$\begin{pmatrix} \Lambda & C \\ 0 & 1 \end{pmatrix}$$

all calculations can be redone in 5D and we will obtain in particular with:

$$\pi = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \\ e \end{pmatrix} \qquad \pi' = \Lambda \pi$$

We can decompose the group Λ in two components, one is orthochron and the other antichron, and write

$$\Lambda = \mu \Lambda_o \quad \text{with} \quad \mu = \pm 1$$

the ANTICHRONS components ($\mu = -1$) invert

- The energy **E**
- The momentum **p**
- The electrical charge **e**

We can express Λ by using the orthochron subset L_0 of the Lorentz group and, by adding ($\lambda = \pm 1$) we introduce (in the two sheets) matter-antimatter duality

$$\Lambda = \begin{pmatrix} \mu L_o & 0 \\ 0 & \lambda \end{pmatrix}$$

The subgroup from the Kaluza group we have chosen is then written

$$\begin{pmatrix} \mu L & 0 & \Delta \xi \\ 0 & \lambda & \Delta \zeta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \xi \\ \zeta \\ 1 \end{pmatrix}$$

APPENDIX 6: IMAGINARY SPACES DO YOU HAVE A SOLE ?

We remember that by by interacting two cosmical subsets of opposite masses and energies, we represented these two sheets like the covering of a projective, which in the case of two dimensions (t,x) became a BOY SURFACE (*)
We also envisioned that the two "poles", one representing the BIG BANG and the other the BIG CRUNCH, instead of being identified, corresponded to a gateway, a bridge linking the two sheets. This made the singularity disappear and moreover, in 2D, gave to the universe-object the topology of a torus T2 arranged in a covering of two sheets of a Klein K2 bottle (more readable in "Topo the world"). The frontier space is then a circle S1

(*) described in details in "Topo the world"

If we place ourselves in 5D we must suppose that we can construct a solution with two metrics of the type

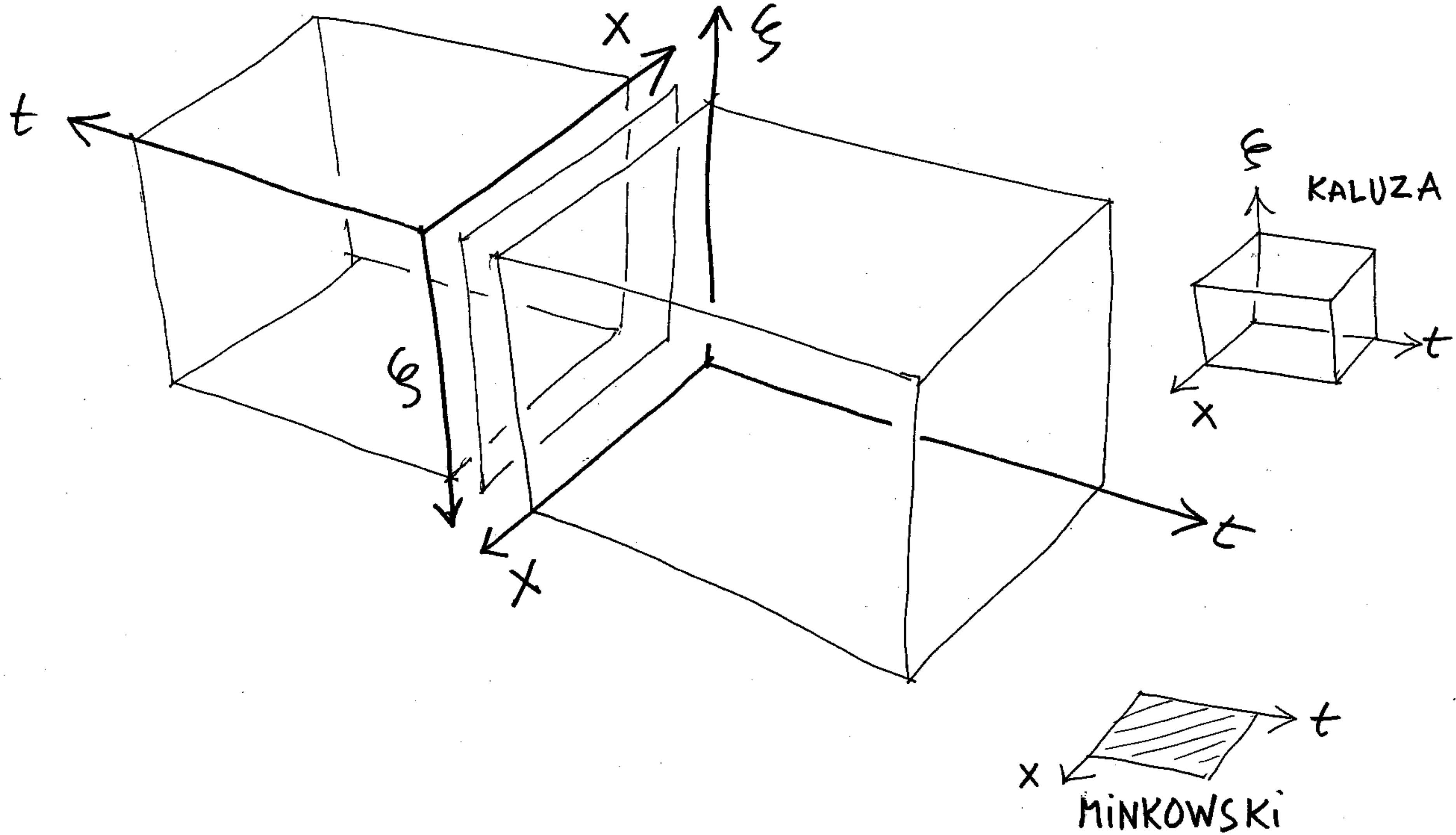
$$\boxed{d\Sigma^2 = R^2 \left[dt^2 - dx^2 - dy^2 - dz^2 - d\zeta^2 \right]}$$

In the primitive Universe (see FASTER THAN LIGHT), before the BREAKING OF SYMMETRY the two scale factors (Warp factors) are supposedly equal. At the juncture, there is a dimensional degeneracy. The metric of the frontier-space then becomes :

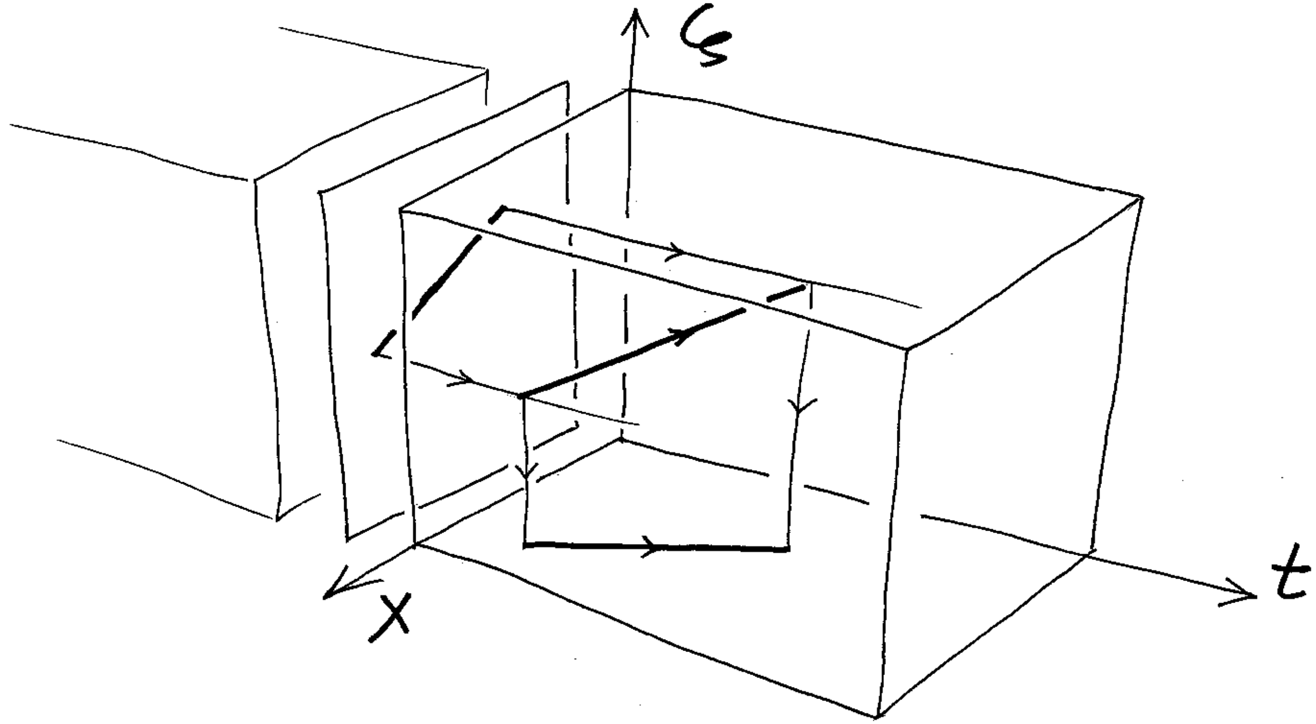
$$\boxed{d\sigma^2 = R_{min}^2 \left[- dx^2 - dy^2 - dz^2 - d\zeta^2 \right] < 0}$$

IN THIS FRONTIER-SPACE, THE LENGTH IS PURE IMAGINARY
CAN IT BE ASSIMILATED TO PURE IMAGINARY TIME ?

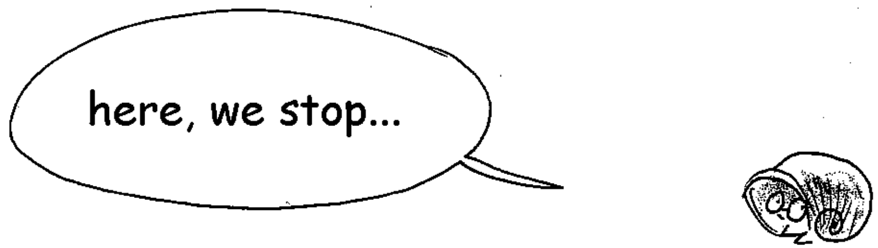
IN ANY CASE, WHAT (META)PHYSICAL SIGNIFICANCE SHOULD WE GIVE TO THIS GEOMETRICAL STRUCTURE ?



the "TOY MODEL"

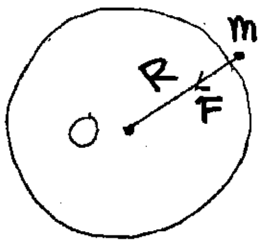


Nobody ever ventured to give some model of what could be the CONSCIENCE with its corollary : CHOICE. Above we have an amusing image where a "line of destiny", achrone, inscribed in this frontier space (x,y,z,ζ) of signature $(- - - -)$ can project itself in an infinity of possible ways in one of the two sheets of spacetime (X,t), the choice of such or such a projection representing a DEGREE OF FREEDOM



APPENDIX 7: NEWTONIAN SOLUTIONS

In 1934, Milne and Mac Crea created a big surprise when, by just using Newton's law and a little bit of calculations, they emerged Friedman's equation, the law of evolution of the characteristic dimension R of the universe. The method consists by considering a small part of the universe, contained in a sphere of radius R centered on O, ρ being the matter density in this sphere.



Then we look what is the acceleration R" to which this mass is submitted by supposing that the point O is fixed. Then we can show that the radial force to which this mass m is submitted is limited to a mass $M = \frac{4}{3}\pi R^3 \rho$ which would be situated in O and which represents the mass contained in this sphere of radius R.

$$F = \frac{-Gm}{R^2}\frac{4}{3}\pi R^3 \rho = m R''$$

we obtain the differential equation :

$$R'' = -\frac{1}{R^2}\left(\frac{4\pi G \rho R^3}{3}\right)$$

If mass is conserved $\rho R^3 = C^{te}$ We obtain Friedman's equation

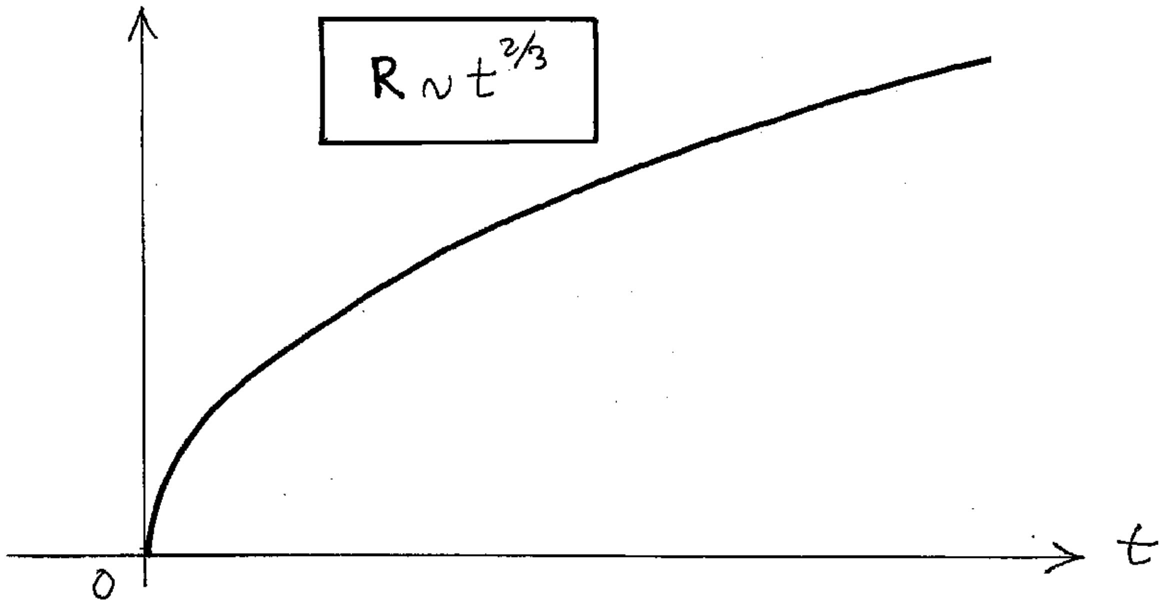
$$R'' = -\frac{a^2}{R^2}$$

which has three types of solutions which all show a deceleration, infinite for R = 0 then decreasing as time increases and R(t) expands. We'll be looking for the law in

$$R \sim t^n$$

$$R' = n a^2 t^{n-1} \quad ; \quad R'' = n(n-1) a^2 t^{n-2} \quad ; \quad R^2 R'' = n(n-1) a^6 t^{3n-2}$$

which leads to the parabolic solution:



Imagine now that the evolution of the Universe is governed by two type of contents, one being positive masses m^+ and the other being negative masses m^- . Moreover, like we tried to make you understand in this comic album, this expansion is being played through two SCALE FACTORS R^+ and R^- (Warp factors)

Let's consider a positive mass m^+ situated on a sphere of radius R^+ which center is assumed to be fixed. Within a Newtonian approximation let's calculate the acceleration $R^{+''}$ that this mass undergoes. It can be calculated by considering, like before, the quantity of positive mass contained in this sphere (and brought back at its center O):

$$\frac{4}{3}\pi \rho^+ {R^+}^3$$

We must take into account of the APPARENT MASS of the negative mass contained in this sphere which is :

$$\frac{4}{3}\pi \rho^- {R^+}^3 \qquad \text{avec} \quad \frac{\rho^-}{\rho^+} = \frac{R^{+3}}{R^{-3}}$$

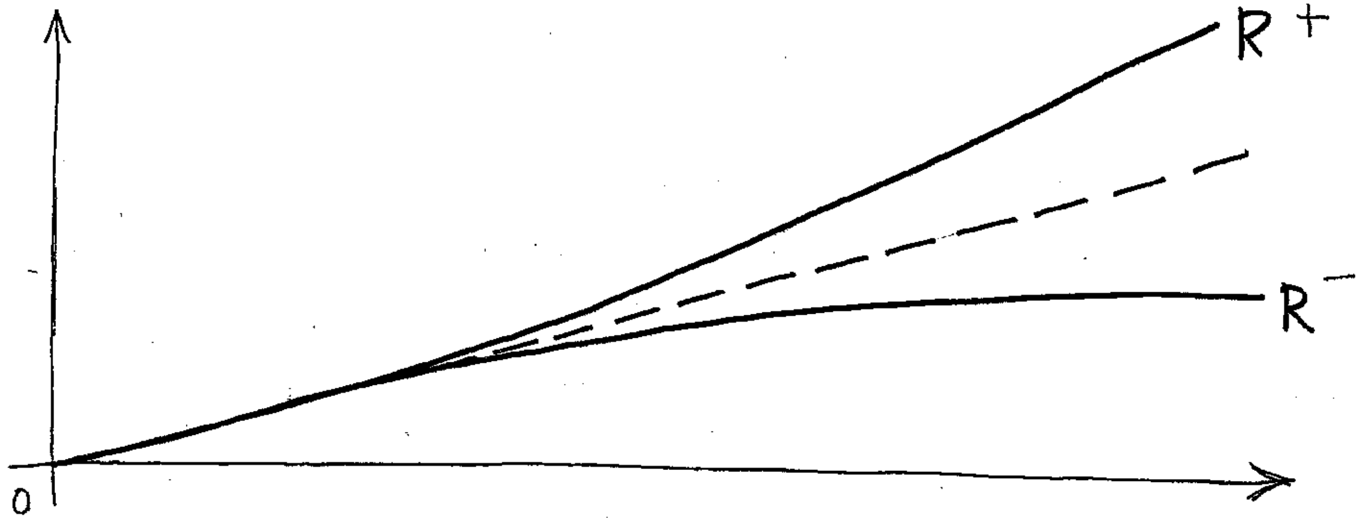
The differential equation giving $R^+(+)$ is then :

$$R^{+''} = -\frac{Gm^+}{R^{+2}} \times \frac{4\pi R^{+3}}{3}\left(\rho^+ - \rho^-\right) = \frac{-a^2}{R^{+2}}\left(1 - \frac{R^{+3}}{R^{-3}}\right)$$

By using the same reasoning and using this time the $R^{-''}$ acceleration undergone by a mass m^- and by taking the constant (arbitrary) equal to 1, we will have this system of two coupled differential equations :

$$\begin{cases} R^{+''} = -\dfrac{1}{(R^+)^2}\left(1 - \dfrac{(R^+)^3}{(R^-)^3}\right) \\[2ex] R^{-''} = -\dfrac{1}{(R^-)^3}\left(1 - \dfrac{(R^-)^3}{(R^+)^3}\right) \end{cases}$$

which allows the linear solution (unstable) $R^+ = R^- \sim t$



The instability of the solution, by supposing that the positive masses undergo a late acceleration will give the illusion of the action of a DARK ENERGY.

These two worlds composed of energies and masses of opposite signs do interact. In the case showed in the previous page, the denser negative masses accelerate the phenomenon of expansion of the positive masses, associated to the scale factor $R^+(+)$. The opposite phenomenon happens in the "negaworld" where observers, composed of negative masses themselves, and receiving signals transported by NEGATIVE ENERGY PHOTONS, would note a deceleration of the expansion phenomenon.

The start of the curve, where expansion seems linear, could seem incompatible with observations. But at this point intervenes a SYMMETRY BREAKDOWN and a VARIATION OF THE CONSTANTS, in particular of the speed of light. Without it the widespread homogeneity of the primitive universe is not explicable. All of this has been discussed in this album:

FASTER THAN LIGHT